

Simple Budgeting Games

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ABSTRACT

Background: We study strategic behavior in participatory budgeting and multiwinner elections based on the Method of Equal Shares (MES). In our model, voters strategically approve projects from a given set, and the mechanism selects a feasible set of projects under MES.

Objectives and Research Questions: We investigate whether Nash Equilibria exist in the resulting class of Simple Budgeting Games, whether such equilibria are easily reached through best-response dynamics, and which fairness properties of MES are preserved when accounting for strategic behavior.

Methods: We take an analytical approach and obtain our results through formal proofs. We present, e.g. examples, constructive procedures, and computational hardness results to prove our claims.

Results: We show that the induced game does not admit a potential function, but a pure-strategy Nash equilibrium always exists. When individual voter budgets are large enough, equilibria guarantee fairness in the sense that each voter has an approved project funded, but it is NP-hard to find an equilibrium in which a given project set is funded. When budgets are too small, every affordable set of projects can be implemented at equilibrium.

Conclusions: Strategic participatory budgeting and multi-winner elections under MES admit pure-strategy Nash equilibria, but there is a sharp qualitative contrast between large and small virtual budgets. We show that deciding the existence or reachability of equilibria with prescribed outcome is computationally intractable. Further, lazy, pivotal behavior has a strategic effect similar to increasing individual budgets.

KEYWORDS

Multiwinner Elections, Participatory Budgeting, Method of Equal Shares, Nash Equilibrium, Best-Response Dynamics

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1 INTRODUCTION

Participatory budgeting (PB) has emerged as one of the most prominent tools of direct democracy, enabling citizens to collectively decide how to allocate portions of public budgets [31, 33]. Since its first use in Porto Alegre in the late 1980s, PB has been adopted by hundreds of cities worldwide, including New York, Paris, and Madrid, as well as by universities, housing associations, and civil society organizations [1, 9, 12, 13]. In these processes, residents submit proposals for local projects—such as renovating a playground, funding cultural events, or improving public transit—and then vote on which projects should be funded subject to a budget constraint.

A fundamental special case of PB arises when all projects have unit cost, which corresponds precisely to the well-established framework of *multiwinner voting (MWV)* [22]. This setting captures scenarios in which projects are relatively homogeneous in cost, such as funding a set of similar community events or neighborhood improvements. While participatory budgeting with heterogeneous project costs presents additional complexity, the unit-cost case (where selecting a committee of k candidates from a larger pool is equivalent to funding k equally-priced projects within a fixed budget) serves as a fundamental building block for understanding fairness and efficiency in collective decision-making. This connection allows fairness concepts and voting rules developed for multiwinner elections, such as *justified representation* and *proportionality*, to be generalized and applied to the broader and more complex context of participatory budgeting where project costs may vary and the number of funded projects is not fixed in advance. Multiwinner elections are particularly important as they form the backbone of representative democracy, from parliamentary elections using proportional representation systems to the selection of corporate boards, academic committees, and jury panels. Moreover, the theoretical insights gained from analyzing this setting often extend naturally to more general participatory budgeting scenarios, making multiwinner elections a natural starting point for developing algorithmic and axiomatic frameworks. In this paper, we follow this suit and first present results for MWV, then generalize them for PB.

A central challenge in PB and MWV is to design a rule that translates voter preferences into a budget allocation that is both fair and easy for participants to understand. One particularly influential method is the *Method of Equal Shares (MES)* [28, 29], which

combines proportional representation of diverse groups with a very transparent cost-sharing interpretation. Under MES, each voter is endowed with an equal share of the budget; projects are funded iteratively if they can be “paid for” by the voters who approve them, with costs distributed proportionally across supporters. MES has attractive axiomatic properties, including proportionality and fairness guarantees, and has been piloted in real-world PB processes.

While MES was designed as a normative rule, its use in practice raises the question of strategic voting. Voters may have an incentive to strategically approve only a subset of the projects they truly like or even (as we will show) approve on the projects that they dislike, in order to maximize the chances that some of their favorite projects are funded. Understanding the strategic properties of MES is crucial for both its theoretical foundations and its practical deployment.

Against this background, in this paper we initiate a systematic study of strategic behavior that arises when self-interested agents participate in budgeting scenarios under MES. We study the method in its “pure” form, without relying on completion rules, which is important for understanding its intrinsic properties. Omitting completion rules also avoids baking in additional normative assumptions, making it easier to identify exactly which outcomes follow from the core mechanism of splitting a common budget (or voting power) equally among voters and using only supporters’ shares to fund projects or elect candidates.

Strategic behavior in budgeting scenarios manifests in various forms: project proposers may manipulate costs to maximize their chances of selection and funding, voters may misrepresent their preferences to achieve more favorable outcomes, and coalitions of agents may coordinate their actions to influence the final selection. We concentrate on scenarios in which voters use approval ballots and may strategically cast approvals that deviate from their true preferences.

We analyze the existence and computational properties of Nash equilibria in what we term *simple budgeting games (SBG)*, investigating how strategic incentives faced by participants affect the fairness properties of the resulting equilibria. In particular, we focus on equilibria that elect a designated set of projects or candidates. Indeed, the outcome of the election is the primary interest of its participants and third parties involved: by looking at the equilibria with a specific winning set we answer the question what stable outcomes can be obtained by strategic agents in such settings, and computational hardness can then be seen as a barrier against control of the outcome in elections with strategic agents.

Our analysis examines the computational complexity of finding equilibria and the fairness of equilibrium outcomes, including how strategic voting may undermine proportionality principles such as justified representation. By studying the interplay between strategic incentives, tractability, and fairness, we provide insights for designing robust budgeting mechanisms that retain desirable properties even under strategic participation. Our contributions are summarized below:

- **Existence of equilibria.** Although the induced SBG does not admit a potential function, we prove that a pure-strategy Nash equilibrium always exists.
- **Fairness and implementability.** We identify a sharp contrast depending on voter budgets:

- With large budgets so that a voter can independently fund some of its approved projects, every equilibrium is fair in the sense that each voter has at least one of its approved projects funded.
- With small individual budgets not covering a project’s cost, every feasible selection of projects is implementable in some equilibrium.

- **Computational hardness.** For large budgets, deciding whether there exists an equilibrium that funds a given set of designated projects, is NP-hard. Moreover, the reachability of such outcomes under *frugal* best-response dynamics where voters change their ballots as little as possible, is NP-hard for both large and small budgets.

- **Lazy voters.** We show that when the electorate includes *lazy* voters [11, 14] who prefer to abstain unless they are pivotal, their presence has a strategic impact that is effectively the same as operating with large budgets, in particular, giving rise to fair Nash equilibria.

The proofs are omitted due to space limitations.

2 RELATED WORK

Participatory budgeting has been widely studied both in political science and in computational social choice [3, 31]. Algorithmic approaches to PB have focused on axiomatic properties (e.g., proportionality, fairness, strategyproofness) and on efficient algorithms for implementing desirable rules. Among these, the Method of Equal Shares [28, 29] has attracted particular attention for its proportional representation guarantees and has been proposed as a canonical rule for approval-based PB.

A special case of PB that has received particular attention in computational social choice is approval-based multiwinner voting, where each of n voters indicates approval or disapproval for every candidate, and a committee of k candidates is selected from a larger candidate set [22]. While the prototypical application of this framework is electing a committee, its applicability extends well beyond traditional electoral contexts [16, 32].

A key criterion in MWV is that the selected committee proportionally reflects the electorate’s preferences [15]. Consequently, recent research has focused on formalizing proportionality notions [3, 7, 19, 24] and devising efficient algorithms to compute proportional committees [4, 6, 28], with a detailed survey of these developments provided by [22].

Most prior work on PB and MWV has assumed sincere votes, but there is growing recognition that strategic voting is a critical concern [3, 5, 8, 10, 20, 21, 23, 26, 27]. The setting most closely related to ours is the class of budgeting games [18] in which central concepts from multiwinner voting theory—such as priceability, the core, and extended justified representation—can be interpreted as equilibrium outcomes. The key distinction from our model lies in the voters’ strategic choices at hand: in budgeting games, voters decide how to distribute a given individual budget across available alternatives, whereas in our setting they strategically choose which projects to approve or disapprove.

Finally, our work relates to the literature on the computational complexity of equilibria in voting and candidacy games [30]. In particular, our NP-hardness result for reaching equilibria echoes

known hardness barriers in candidacy games, underscoring the difficulty of predicting outcomes under strategic participation.

3 MODEL AND PRELIMINARIES

In this section, we formalize the notions and concepts relevant for our analysis.

3.1 Simple budgeting games

A *participatory budgeting (PB)* instance is given as a tuple $I = (\mathcal{P}, c, b, \mathcal{V}, \mathbf{A})$ where

- $\mathcal{P} = \{p_1, \dots, p_m\}$ is the set of *projects* (or, *candidates*),
- $c : \mathcal{P} \rightarrow \mathbb{R}_{>0}$ is the *cost function*, assigning to each project $p \in \mathcal{P}$ its cost $c(p) \in \mathbb{R}_{>0}$,
- $b \in \mathbb{R}_{>0}$ is the *budget limit*,
- $\mathcal{V} = \{1, \dots, n\}$ is the set of *voters* (or, *agents*),
- $\mathbf{A} = (A_1, \dots, A_n)$ is the *preference profile*, where $A_i \subseteq \mathcal{P}$ is voter i 's set of *approved projects*.

For any subset of projects $P \subseteq \mathcal{P}$, let $c(P) = \sum_{p \in P} c(p)$ be its total cost. A subset of projects $P \subseteq \mathcal{P}$ is *feasible* if $c(P) \leq b$. It is assumed that all individual projects are feasible, i.e., $c(p) \leq b$ for all $p \in \mathcal{P}$. An instance is said to have *unit costs* if for every project $p \in \mathcal{P}$, we have $c(p) = 1$ and $b \in \mathbb{N}_{>0}$; these instances correspond to *multiwinner voting (MWV)*.

When facing an instance $I = (\mathcal{P}, c, b, \mathcal{V}, \mathbf{A})$, each voter $i \in \mathcal{V}$ strategically submits a ballot $B_i \subseteq \mathcal{P}$ which may or may not coincide with its truthful preferences A_i over the projects in \mathcal{P} . The vector $\mathbf{B} = (B_1, \dots, B_n)$ of such ballots (i.e., votes) is called a *voting profile*.

The outcome of the instance I , when voters cast voting profile \mathbf{B} , is a set of funded projects $W \subseteq \mathcal{P}$ such that $c(W) \leq b$. The set W is determined using a *PB rule*, which is a function that takes as input the tuple $(\mathcal{P}, c, b, \mathcal{V}, \mathbf{B})$ (which is exactly I if voters submit their true preferences, i.e., if $\mathbf{B} = \mathbf{A}$) and returns the set of funded projects $W \subseteq \mathcal{P}$. In the context of MWV, the outcome of the process is called the (winning) *committee*; we shall also refer to it as the *winning set*.

The utility of agent $i \in \mathcal{V}$ from an outcome $W \subseteq \mathcal{P}$ is given by the number of its truly approved projects in the winning set: $u_i(W) = |W \cap A_i|$. Coupled with the above ingredients, this induces a normal form game among the voters, which we term the *simple budgeting game (SBG)*.

3.2 Method of equal shares

The PB rule in the focus of our analysis is the *Method of Equal Shares (MES)* [28]. MES works in rounds. At the beginning, it virtually allocates each voter an equal part of the budget, $b_i(1) = b/n$. This quantity serves as “virtual money” and will be referred to as a voter’s *individual virtual budget*. The process starts with an empty committee $W = \emptyset$, and in each round the method adds to W one project according to the following procedure:

- (1) For each project $p \in \mathcal{P}$ that has not yet been selected, the method attempts to distribute the project’s cost proportionally based on the ballots submitted by the voters, while considering that some voters may have already exhausted their budgets. Formally, let $b_i(t)$ be the amount of virtual money

that voter i is left with just before the start of the t 'th iteration. In the t 'th round, the rule selects the candidate that should be added to W as follows. For a value $\rho \geq 0$, we say that a project $p \notin W$ is ρ -affordable at step t if

$$\sum_{i \in B(p)} \min(\rho, b_i(t)) \geq c(p) \quad (1)$$

where $B(p)$ is the set of voters who vote for p . Intuitively, if a project p is ρ -affordable then, the cost of the project can be spread among the voters who support it in a way that each voter pays at most ρ .

- (2) If no candidate $p \notin W$ is ρ -affordable for any ρ , the rule stops and returns W . This happens when for each unselected project p the remaining amount of money in the virtual accounts of those voters who submitted a positive ballot on p is lower than its cost: $\sum_{i \in B(p)} b_i(t) < c(p)$.
- (3) Otherwise, the method selects the project $p \notin W$ that is ρ -affordable for the lowest value of ρ —i.e., the project that minimizes the amount that the voters need to pay—and adds it to committee W . If there are several such projects, we break the ties lexicographically. Note that by the minimality of ρ , (1) holds with equality. The voters’ budgets are updated accordingly: for each voter $i \in B(p)$, we set its budget to $b_i(t+1) = b_i(t) - \min(\rho, b_i(t))$ (so each of these voters pays either ρ or its entire remaining budget for p); for each $i \notin B(p)$, we set $b_i(t+1) = b_i(t)$.

3.3 Justified representation

The Method of Equal Shares is a key rule in multiwinner voting and participatory budgeting due to its strong proportional representation properties. It ensures that cohesive groups of voters—those who collectively approve a sufficient share of the budget—are guaranteed a fair level of representation in the outcome. In particular, MES satisfies *Extended Justified Representation (EJR)*, a strong variant of *Justified Representation (JR)* [3].

A committee W satisfies justified representation if no sufficiently large and cohesive group of voters is left completely unrepresented. Formally, W satisfies JR if and only if:

$$\left(\forall V \subseteq \mathcal{V} : |V| \geq \frac{n}{b} \text{ and } \bigcap_{i \in V} A_i \neq \emptyset \right) \Rightarrow W \cap \bigcup_{i \in V} A_i \neq \emptyset.$$

That is, if there exists a group V of at least n/b voters who all approve at least one common candidate, then at least one voter in the group must have an approved candidate in the winning committee. This ensures that a group large enough to “deserve” one seat is not “entirely ignored”.

EJR is a stronger property than JR. It ensures that large cohesive groups receive proportional representation, rather than just a single representative. A committee W satisfies EJR if and only if:

$$\left(\forall l \in \{1, \dots, b\} : \forall V \subseteq \mathcal{V} : |V| \geq l \cdot \frac{n}{b} \text{ and } \left| \bigcap_{i \in V} A_i \right| \geq l \right) \Rightarrow \exists i \in V \text{ s.t. } |W \cap A_i| \geq l.$$

Intuitively, if there is a group V large enough that, proportionally, they “deserve” l winners, and this group collectively approves at least l common candidates, then at least one voter in the group

must have at least l of its approved candidates elected. This prevents situations where a large group receives too few winners, ensuring stronger proportionality than JR.

Proportional Justified Representation (PJR) occupies an intermediate position between JR and EJR. A committee W satisfies PJR if, for every integer l with $1 \leq l \leq b$, no sufficiently large and cohesive group is under-represented:

$$\left(\forall l \in \{1, \dots, b\} : \forall V \subseteq \mathcal{V} : |V| \geq l \cdot \frac{n}{b} \text{ and } \left| \bigcup_{i \in V} A_i \right| \geq l \right) \Rightarrow \left| W \cap \bigcup_{i \in V} A_i \right| \geq l.$$

Intuitively, if there is a group V large enough that, proportionally, they “deserve” l seats, and this group collectively approves at least l distinct candidates, then at least l candidates among those approved by at least one voter in the group must be in the winning set. This ensures that large cohesive groups receive collective proportional representation, even if individual voters may not each get all their preferences met.

We study how robust the proportionality guarantees of the method of equal shares are when voters behave strategically rather than truthfully. While MES is known to provide strong proportional representation under sincere approvals (such as EJR), these guarantees may fail once voters have incentives to misreport. To investigate this, we examine simple budgeting games in which voters strategically choose which projects to approve to maximize their individual utility, and ask whether MES still delivers proportional outcomes or whether manipulation can significantly undermine its fairness. This sheds light on the practical applicability of MES in real-world scenarios, where strategic behavior is hard to avoid.

4 STRATEGIC BEHAVIOR

In this section, we identify several important characteristics of simple budgeting games. In particular, we show that voters may benefit from strategically misreporting their preferences, either by under-reporting or over-reporting relative to their truthful ballots. Surprisingly, in one of our examples, over-reporting emerges as the only profitable deviation available to voters. Finally, we demonstrate that a sequence of improving deviations can form a cycle, thereby establishing that no *potential function* exists for this class of games.

A (generalized ordinal) potential function [25] for a game with players $\mathcal{V} = \{1, \dots, n\}$ having strategy sets S_1, \dots, S_n and utility functions u_1, \dots, u_n , is a function $\phi : \times_{i=1}^n S_i \rightarrow \mathbb{R}$ that satisfies the following for every $i \in \mathcal{V}$, every $s \in \times_{i=1}^n S_i$, and every $s'_i \in S_i$:

$$u_i(s'_i, s_{-i}) > u_i(s) \Rightarrow \Phi(s'_i, s_{-i}) > \Phi(s),$$

where (s'_i, s_{-i}) denotes the strategy profile obtained from s by replacing i 's strategy with s'_i . A potential function would imply existence of a pure-strategy Nash equilibrium (PSNE), something that we will thus have to prove separately (which we do in Section 5 below). The non-existence of a potential function furthermore means that *better response dynamics* (the process of agents iteratively changing their strategies so as to improve their own utility) may cycle among a set of strategy profiles, and does not always converge to a PSNE.

REMARK 1. We provide examples (Examples 4.1– 4.3) for MWV instances, both for the ease of exposition and to indicate the susceptibility of this subclass of simple budgeting games to strategic voter behavior with cycling dynamics. Note though, that all the results in this section are easily transformable to a general PB setting, and so apply to all SBGs.

We start with an example that shows that truthful reporting is not, in general, a pure-strategy Nash equilibrium

Example 4.1. Consider a simple budgeting game with n voters and $n + 1$ projects with unit costs. We let $b = n$, so each agent has a virtual budget of 1. For a voter $i \in \{1, 2, \dots, n - 2\}$, the preference set is given by $A_i = \{p_i, p_{n-2}\}$, so that all these voters approve of the candidate that they are indexed to as well as the candidate indexed at $n - 2$. For voter $n - 1$ we have $A_{n-1} = \{p_{n-2}, p_{n-1}, p_n\}$, and the approval set of voter n is given as $A_n = \{p_{n-1}, p_n, p_{n+1}\}$.

Initially, every voter votes truthfully, in which case MES will elect the committee $W = \{p_{n-2}, p_{n-1}\}$, resulting in utility $u_{n-1}(W) = |W \cap A_{n-1}| = 2$ for voter $n - 1$.

We now let voter $n - 1$ to deviate from its initial truthful vote to a strategy where it approves only on projects $\{p_{n-1}, p_n\}$. MES will now elect the committee $W = \{p_{n-2}, p_{n-1}, p_n\}$, and the utility of voter $n - 1$ will grow to $u_{n-1}(W) = |W \cap A_{n-1}| = 3$.

From the example above, we observe that a voter can under-report its approvals to increase its own utility. This strategy is not only profitable for the voter but also results in a PSNE. However, under-reporting is not the only form of strategic deviation available. Voters may also over-report their approvals, and while it may seem counterintuitive that doing so could improve a voter's utility, we demonstrate that over-reporting can indeed be beneficial and can likewise lead to a Nash Equilibrium.

Example 4.2. Consider a simple budgeting game with 9 projects $\{p_1, \dots, p_9\}$ with unit costs and 10 voters, whose preferences over the projects are shown in Table 1. As in the previous example, all voters have a virtual budget of 1, and we assume lexicographic tie-breaking.

Table 1: Preference profile = initial voting profile in Example 4.2

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
1	1	1	1					
2	2	2		2				
3	3	3		3				
			4					
			5					
				6	6			
				7	7	7	7	
				8	8	8	8	
				9	9	9	9	
						10	10	10

Initially, all voters vote truthfully, and MES will elect the committee $W = \{p_6, p_7, p_8, p_9, p_1, p_2, p_3\}$, resulting in utility $u_6(W) = |W \cap A_6| = 1$ for voter 6.

Now, if 6 adds project p_4 to its vote, then the sequence of funded projects will be $p_4 \rightarrow p_6 \rightarrow p_7 \rightarrow p_8 \rightarrow p_9 \rightarrow p_1 \rightarrow p_2 \rightarrow p_5$, resulting in utility $u_6(W) = |W \cap A_6| = 2$ for voter 6.

In contrast to Examples 4.1 and 4.2, the next example shows that best-response dynamics can form a cycle, where voters repeatedly alternate between profitable deviations without ever reaching a stable outcome, i.e., a Nash equilibrium. This cyclical behavior demonstrates that no potential function can exist for simple budgeting games, since the presence of such a function would guarantee that every improving deviation leads to monotonic progress toward equilibrium. The example thus provides a direct witness of the structural obstacles to convergence to equilibrium in these games.

Example 4.3. Let Γ be a simple budgeting game with 6 projects $\{p_1, \dots, p_6\}$ with unit costs and 4 voters, whose preferences over the candidates are shown in Table 2. As in the previous example, all voters have a virtual budget of 1, and we assume lexicographic tie-breaking. The initial voting profile with its respective winning set of projects and a cycle of best responses are shown in Table 3: The first row represents an initial voting profile, and each subsequent row represents the voting profile resulting from a best response of a player. The initial profile, in turn, is obtained as a best response by a player from the profile corresponding to the last row of the table, thus forming a cycle.

Table 2: Preference profile for Example 4.3

p_1	p_2	p_3	p_4	p_5	p_6
				1	
2	2	2	2		
				3	3
					4

Table 3: Voting profiles and corresponding winning sets along a cycle of best responses in Example 4.3

p_1	p_2	p_3	p_4	p_5	p_6	W
1,2	1,2			1,3	3,4	$W = \{p_1, p_2, p_6\}$
2	2	1	1	1,3	3,4	$W = \{p_1, p_5, p_6\}$
		1,2	1,2	1,3	3,4	$W = \{p_3, p_4, p_6\}$
1	1	2	2	1,3	3,4	$W = \{p_3, p_5, p_6\}$
1,2	1,2			1,3	3,4	$W = \{p_1, p_2, p_6\}$

Along the cycle, Voters 3 and 4 vote truthfully, and Voters 1 and 2 strategize. In the initial voting profile, Voter 1 votes for projects $\{p_1, p_2, p_5\}$ and has utility 0 as its only favorite project p_5 is not elected. Voter 1 then moves to vote for projects $\{p_3, p_4, p_5\}$ and increases its utility to 1. The utility of Voter 2 who votes for projects $\{p_1, p_2\}$ at this step is currently 1. Now, Voter 2 moves to vote for projects $\{p_3, p_4\}$ instead and increases its utility to 2. This move, however, removes the project p_5 from the winning selection, and so Voter 1 has utility 0 again. Now, Voter 1 returns to its original

strategy $\{p_1, p_2, p_5\}$ and increases its utility to 1, which makes the utility of Voter 2 drop from 2 to 1. As a result, Voter 2 also returns to its original strategy $\{p_1, p_2\}$, which raises its utility back to 2.

Despite the non-existence of a potential function in simple budgeting games, the question of whether a pure-strategy Nash equilibrium exists remains both natural and important. Equally relevant is the issue of reachability: while arbitrary better-response dynamics do not converge in this setting, one may still ask whether convergence can be ensured under more restricted forms of better- or best-response dynamics. These considerations motivate our analysis in the following sections, where we investigate both the existence of equilibria and the conditions under which they can be reached through structured strategic behavior.

5 NASH EQUILIBRIUM

In this section, we show the existence and properties of PSNE in simple budgeting games. We distinguish between the cases with *small* virtual budgets where $b/n < c(p)$ for all $p \in \mathcal{P}$ and *large* budgets where there is a voter $i \in \mathcal{V}$ with $b/n \geq c(p)$ for some $p \in A_i$. Note that in the MWV case, this corresponds to $b < n$ and $b \geq n$, respectively. In addition, we separate between the settings with *basic* voters in the original definition of SBG and *lazy* voters who gain additional utility $0 < \epsilon < 1$ from abstaining, so they would abstain whenever they are not pivotal [14].

5.1 Existence

We prove the existence of PSNE in simple budgeting games with small and large budgets in settings with basic and lazy voters. For basic voters, finding equilibria with a given outcome is polynomial-time solvable for small budgets and NP-hard for large budgets; for lazy voters, the problem becomes NP-hard already for small budgets.

5.1.1 Basic voters. We start with the following theorem.

THEOREM 5.1. *Let Γ be a simple budgeting game with basic voters and small virtual budgets. Then, for any affordable set of projects $Q \subseteq \mathcal{P}$, $c(Q) \leq b$, there exists a pure-strategy Nash equilibrium where all the projects in Q are funded.*

Unlike in the case with small individual voter budgets where any affordable set of projects is part of the winning set for some PSNE, in the case with large budgets it is NP-hard to decide whether a given set of projects can be funded in an equilibrium (Theorem 5.2). Our proof makes use of the EXACT 3-COVER (X3C) problem. An instance of this problem is given by a set of $3r$ ground elements $A = \{a_1, \dots, a_{3r}\}$, for some $r \in \mathbb{N}$, and a family $\mathcal{Z} = \{Z_1, \dots, Z_q\}$ of 3-element subsets of A . Let $Z_\ell = \{a_{i_\ell}, a_{j_\ell}, a_{k_\ell}\}$ for some $i_\ell, j_\ell, k_\ell \in \{1, \dots, 3r\}$. It is a “yes”-instance if there exists a subfamily $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ such that $\cup_{Z \in \tilde{\mathcal{Z}}} Z = A$ and $Z_i \cap Z_j = \emptyset$ for all $Z_i, Z_j \in \tilde{\mathcal{Z}}$ with $i \neq j$; otherwise, it is a “no”-instance. We additionally assume that $r \geq 2$ and that each element of A is contained in exactly three distinct sets in \mathcal{Z} ; note that this implies that $q = 3r$. This restricted variant of X3C, which we shall refer to as RX3C, is NP-complete [17].

THEOREM 5.2. *Let Γ be a simple budgeting game with basic voters and large virtual budgets, and let $Q \subseteq \mathcal{P}$ be a subset of designated*

projects. Then, it is NP-hard to decide whether there is a PSNE with a winning set W such that $Q \subseteq W$.

Nonetheless, we can prove the existence of a pure-strategy Nash equilibrium also for the games with large budgets.

THEOREM 5.3. *Let Γ be a simple budgeting game with basic voters and large virtual budgets. Then, Γ has a PSNE.*

5.1.2 Lazy voters. In the presence of lazy voters, the statement of Theorem 5.1 no longer holds as stated. Analogously to the case of large budgets, introducing lazy voters refines the set of Nash equilibria by ruling out some profiles that were equilibria under basic voters. At the same time, this restriction does not destroy equilibrium existence: one can still show that pure-strategy Nash equilibria always exist even when some voters are lazy.

THEOREM 5.4. *Let Γ be a SBG with lazy voters. Then:*

- (1) *For small virtual budgets, the profile in which all the voters abstain is always an equilibrium.*
- (2) *For small virtual budgets, if $\exists p \in \mathcal{P}$ such that $b/n|A(p)| \geq c(p)$ where $A(p)$ is the set of voters who approve of project p , then there is a PSNE with a non-empty set of active (i.e. non-abstaining) voters. Otherwise, the equilibrium where everyone abstains is unique.*
- (3) *For large budgets, there is always a PSNE with active voters, but no equilibrium with all abstaining.*

We now turn to PSNE with a fixed winning set. Whereas in SBGs with basic voters the existence of such equilibria is trivial for small virtual budgets and NP-hard to decide for large budgets, we show that once lazy voters are present, the problem becomes computationally hard for all budget sizes.

THEOREM 5.5. *Let Γ be a simple budgeting game with lazy voters, and let $Q \subseteq \mathcal{P}$ be a subset of designated projects. Then, it is NP-hard to decide whether there is a PSNE with a winning set W such that $Q \subseteq W$.*

5.2 Fairness properties

By Theorem 5.1, since any affordable set of projects can be sustained as an equilibrium outcome of a SBG when the virtual budgets are small, the fairness guarantees of the method of equal shares do not necessarily carry over when voters behave strategically. In particular, when the game is played with small virtual budgets, the strategic incentives of voters may allow equilibria that fail to reflect the proportional representation properties such as EJR, or even JR, that MES ensures under truthful participation. This indicates a crucial gap between the theoretical fairness guarantees of MES in the truthful case and the practical outcomes that may arise when voters act strategically in budgeting contexts. However, as we show below, increasing the budget or introducing lazy voters to the game can restore the justified representation, albeit not its stronger variants.

For the following result, we need to strengthen the definition of large budgets as follows. We say that a virtual budget is *super-large* if the requirement that $b/n \geq c(p)$ for some $p \in A_i$ holds for all voters $i \in \mathcal{V}$. Note that for multiwinner voting instances, the definitions of large and super-large budgets coincide. We have the following lemma.

LEMMA 5.6. *Let Γ be a SBG with basic voters and super-large virtual budgets. Then, for any equilibrium winning set W and voter i , either $A_i \cap W \neq \emptyset$ or $A_i = \emptyset$.*

We thus see that unlike the case with small virtual budgets, in simple budgeting games with super-large budgets, every equilibrium outcome is guaranteed to satisfy JR, ensuring that no sufficiently large cohesive group of voters is left entirely without representation. However, the stronger proportionality guarantees of MES under truthful participation do not carry over: EJR is no longer satisfied in equilibrium. Moreover, as we demonstrate in the following Example 5.7, even PJR can fail to hold, showing that equilibria under strategic behavior may fall significantly short of the proportional fairness ensured by truthful MES. The example further shows that the equilibrium project set may be as small as roughly half the number of voters. This outcome may appear counterintuitive, since Lemma 5.6 could be interpreted as suggesting that equilibrium winning sets should necessarily include a large number of projects—potentially as many as the number of voters.

Example 5.7. Let Γ be a simple budgeting game with $b = n$ and unit costs, and with $m = n = 2q$ projects and basic voters. We show an example of PSNE with $q + 1$ funded projects. For $q \rightarrow \infty$ we have $(q + 1)/2q \rightarrow 1/2$.

Table 4: Voting profile for $q = 3$ in Example 5.7

p_1	p_2	p_3	p_4	p_5	p_6
1,2	1	2	3		
3,4	4	5	6		
5,6					

The voting profile for $q = 3$ is shown in Table 4. The preference profile is such that every voter approves all the projects. The only potential deviation would be for a voter to withdraw all current approvals and instead support only the unfunded projects. However, such a deviation would not increase the voter's utility, since its corresponding project p_2, p_3 or p_4 would then fail to be funded. Hence, the outcome constitutes a Nash equilibrium.

It is important to note, though, that the group of all six voters is 6-cohesive and, under PJR, should be entitled to six winners, whereas the equilibrium outcome contains only four.

Note that for lazy voters, the statement of Lemma 5.6 holds regardless of budget size for all *active* (i.e., non-abstaining) voters, as they are pivotal by definition.

LEMMA 5.8. *Let Γ be a simple budgeting game with lazy voters. Then, for any equilibrium winning set W and active voter i , there is a project $p \in A_i \cap W$ for which i is pivotal.*

Coupled with the arguments in the proof of Lemma 5.6, the above observation implies the following lemma.

LEMMA 5.9. *Let Γ be a SBG with lazy voters and super-large virtual budgets. Then, for any equilibrium winning set W and any voter i (whether active or not), either $A_i \cap W \neq \emptyset$ or $A_i = \emptyset$. For each active voter i , there is a project $p \in A_i \cap W$ for which i is pivotal.*

By Lemma 5.8, in the case where voters gain additional utility from abstaining, active voters get represented in an equilibrium of a simple budgeting game, regardless of the budget size. This is in contrast with the basic case of the game with small budgets where any subset of alternatives may arise as a game outcome, and so none of the voters are guaranteed justified representation. While Lemmas 5.6 and 5.9 imply JR for both basic and lazy SBG settings, the following Example 5.10 demonstrates that the presence of lazy voters refines the set of PSNE even further than increasing the budget. Moreover, the example also justifies the necessity of super-large budget definition in these two lemmas.

Example 5.10. Consider a simple budgeting game Γ with voters $\mathcal{V} = \{a, b, c\}$ and projects $\mathcal{P} = \{p_1, p_2, p_3, p_4\}$. The costs of the projects are $c(p_1) = c(p_2) = 1$ and $c(p_3) = c(p_4) = 4$, and the total budget is $b = 4$ (so $b/n = 4/3$). The approval sets are given by $A_a = \{p_1\}$, $A_b = \{p_2\}$ and $A_c = \{p_4\}$.

With basic voters, the strategy profile where $B_a = \{p_1, p_3\}$, $B_b = \{p_2, p_3\}$ and $B_c = \{p_3\}$ is a Nash equilibrium with $W = \{p_1, p_2, p_3\}$. However, voter c is not represented in this solution as its favorite project p_4 is not chosen.

With lazy voters, the above profile will not be an equilibrium, as c would prefer to abstain. Instead, we shall have a PSNE with $B_a = \{p_1\}$, $B_b = \{p_2\}$ and $W = \{p_1, p_2\}$, where c is abstaining and the active voters a and b are both represented.

6 FRUGAL BEST RESPONSES

In this section, we turn our attention to the reachability of pure-strategy Nash equilibria in simple budgeting games under restricted best-response dynamics. Since arbitrary best-response dynamics may fail to converge, we introduce a more disciplined variant that captures behavior where voters tend to exert minimal effort into changing their ballot, which we term *frugal best responses*. The key idea is that voters deviate only when doing so provides a clear utility gain with the smallest possible change to their ballot, thereby avoiding unnecessary oscillations.¹ Formally, let $\text{BR}_i(\mathbf{B})$ be the set of best responses of agent i in voting profile \mathbf{B} , i.e. $\text{BR}_i(\mathbf{B}) = \arg_{B'_i} \max \{u_i(B'_i, B_{-i}) : B'_i \subseteq \mathcal{P}\}$. The frugal best responses are those that minimize the symmetric difference with the current strategy B_i : $\text{FBR}_i(\mathbf{B}) \subseteq \arg_{B'_i} \min \{|B'_i \Delta B_i| : B'_i \in \text{BR}_i(\mathbf{B})\}$.

We now focus on the question of reachability of a PSNE with a prescribed set of funded projects under frugal best-response dynamics. While the existence of such equilibria is straightforward in the case of small virtual budgets and shown to be NP-hard to decide for large budgets in SBGs, we establish that the reachability problem is computationally hard across the board. In particular, we prove that determining whether a given set of projects can be reached as the outcome of frugal best-response dynamics is NP-hard even for small virtual budgets, highlighting a fundamental complexity barrier that persists regardless of budget size.

THEOREM 6.1. *Let Γ be a simple budgeting game with basic voters, and let $Q \subseteq \mathcal{P}$ be a subset of designated projects. Then, from a given voting profile, it is NP-hard to decide whether there is a sequence of*

¹The definition follows the idea behind *lazy* best-response dynamics in the context of congestion games [2] but is simpler than theirs. We choose to use another name to also differentiate it from the notion of *lazy voters*.

frugal best responses that leads to a PSNE with a winning set W such that $Q \subseteq W$.

Our hardness reduction covers the case where the starting point of improving dynamics is composed of truthful but not necessarily complete ballots (that is, voters vote only for their approved projects but not necessarily all of them), which is realistic as the voters may choose not to spread their budget among too many project but increase the chances for funding of fewer but important ones. Nonetheless, we believe that our hardness result extends to the complete truthful case as well, albeit the reduction will be more complicated.

7 CONCLUSION

In this paper, we examined strategic behavior in participatory budgeting and multiwinner elections under the Method of Equal Shares, modeling voters as strategic agents who choose approval ballots in a simple budgeting game. Using this framework, we established the guaranteed existence of pure-strategy Nash equilibria, identified a sharp contrast between large and small virtual budgets, and showed that deciding the existence or reachability of equilibria with prescribed outcome is computationally intractable in several natural variants of the game. The results further revealed that the presence of lazy, pivotal-only voters has a strategic effect similar to increasing individual budgets, partially restoring fairness guarantees while preserving complexity barriers.

However, our analysis leaves several natural questions open for future work. In particular, the convergence of frugal best-response dynamics to a Nash equilibrium remains unresolved (though we conjecture that it always succeeds). While we focused on the Method of Equal Shares due to its prominence and strong proportionality guarantees in multiwinner voting and participatory budgeting, it would be valuable to extend the analysis to other rules and compare their strategic properties to MES. Furthermore, the impact of different completion rules on equilibrium structure and fairness warrants further investigation. Finally, exploring richer solution concepts, such as strong equilibria, could provide additional insights into the robustness of these mechanisms under more demanding stability notions.

ETHICAL STATEMENT

This paper presents purely theoretical work on strategic behavior in participatory budgeting and multiwinner elections under the Method of Equal Shares. It involves no human subjects, personal data, real-world experiments, or empirical studies of actual elections. The analysis aims to advance understanding of algorithmic game theory and computational social choice for fair collective decision-making, and the computational results are not intended as prescriptive guidance for manipulating electoral outcomes.

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