

# Is Four Enough? Automated Reasoning Approaches and Dual Bounds for Condorcet Dimensions of Elections

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## ABSTRACT

**Background:** In an election where  $n$  voters rank  $m$  candidates, a *Condorcet winning set* is a committee of  $k$  candidates such that for any outside candidate, a majority of voters prefer some committee member. Condorcet’s paradox shows that some elections admit no Condorcet winning sets with a single candidate (*i.e.*,  $k = 1$ ), and the same can be shown for  $k = 2$ . On the other hand, recent work proves that a set of size  $k = 5$  exists for every election. This leaves an important theoretical gap between the best known lower bound ( $k \geq 3$ ) and upper bound ( $k \leq 5$ ) for the number of candidates needed to guarantee existence.

**Objectives and Research Questions:** We aim to close the gap between the existence guarantees and impossibility results for Condorcet winning sets. We explore the use of an automated reasoning approach to tighten these upper and lower bounds.

**Methods:** We design a *mixed-integer linear program (MILP)* to search for elections that would serve as counter-examples to conjectured bounds. We employ a number of optimizations—such as symmetry breaking, subsampling, and constraint generation—to enhance the search and model effectively infinite electorates. Furthermore, we analyze the dual of the linear programming relaxation as a path towards obtaining a new upper bound.

**Results:** Despite extensive search on moderate-sized elections, we fail to find any election requiring a committee larger than size 3. Motivated by our experimental results in this direction, we simplify the dual linear program and formulate a conjecture which, if true, implies that a winning set of size 4 always exists.

**Conclusions:** Our automated reasoning results provide strong empirical evidence that the Condorcet dimension of any election may be smaller than currently known upper bounds, at least for small instances. We offer a general-purpose framework for searching elections in ranked voting and a new, concrete analytical path via duality toward proving that smaller committees suffice.

## KEYWORDS

Voting Theory, Condorcet Winning Sets, Automated Reasoning

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## 1 INTRODUCTION

Voting theory provides a formal framework for studying principled methods of aggregating individual preferences into collective decisions. A central challenge in designing robust voting mechanisms is *Condorcet’s paradox*: in an election where voters have ranked preferences, the will of the majority can be cyclical. As illustrated in Table 1, it is possible that for *any* single candidate chosen as the winner, a vast majority of voters would prefer an alternative candidate. In the worst-case, the fraction of voters that prefer the alternative candidate over the selected winner can come arbitrarily close to 1, rendering single-winner systems theoretically unstable.

Motivated by this paradox, Elkind et al. [9] propose a relaxation of the single-winner assumption of ranked voting; they define a winning *committee* of  $k$  candidates. This relaxation is natural for many applications, such as electing parliamentary bodies, hiring committees, or shortlists for awards. They termed such a committee a *Condorcet winning set (CWS)* if no candidate outside of the committee is preferred to all committee members by a majority of the voters. The core question in this domain is identifying the *Condorcet dimension*: what is the smallest size of a committee needed to guarantee that a CWS of that size exists in any election?

More generally, we consider the setting where the majority threshold ( $1/2$ ) is replaced by an arbitrary  $\alpha$ :

A committee  $S$  is  $\alpha$ -undominated if for all candidates  $a \notin S$ , the fraction of voters preferring  $a$  to every member of  $S$  is strictly less than  $\alpha$ . For what values of  $k \in \mathbb{Z}^+$  and  $\alpha \in (0, 1]$  does every election have an  $\alpha$ -undominated committee of size  $k$ ?

Despite significant progress, there remains a gap between the theoretical lower and upper bounds for the Condorcet dimension. Elkind et al. [9] construct specific elections that require a committee of size  $k = 3$  (for  $\alpha = 1/2$ ). Generalizing this, Charikar et al. [8] show

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$v_1$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$
1	2	3	4	5	6
2	3	4	5	6	1
3	4	5	6	1	2
4	5	6	1	2	3
5	6	1	2	3	4
6	1	2	3	4	5

**Table 1: An election showing that for any candidate chosen as the winner, there are 5/6 voters who prefer some other candidate.**

that if a size- $k$  committee is always guaranteed, then  $\alpha$  must be at least  $\frac{2}{k+1}$ . On the other side of the bound, early work showed that a logarithmic size set of candidates always suffices [9]. In a recent breakthrough, Charikar et al. [8] prove that a set of size  $k = 6$  is always sufficient. For general  $\alpha$ , they establish that if  $\frac{\alpha}{1-\ln \alpha} \geq \frac{2}{k+1}$ , a set exists. Most recently, Nguyen et al. [16] refine this analysis to show that  $k = 5$  candidates always suffices. This leaves us with a significant gap. We know that committees of size 2 are insufficient, yet committees of size 5 are always sufficient.

## 1.1 Our contributions

In this work, we utilize an automated reasoning approach to attempt to tighten these bounds. We make the following contributions.

We design a *mixed integer linear program (MILP)* to search for ranked voting elections with large Condorcet dimensions. While the problem scales exponentially with the size of the election, we implement optimizations such as symmetry breaking to search the space of elections with up to  $m = 9$  candidates. Using structural abstractions, we construct an enhanced MILP to search over structured elections with effectively infinite candidates.

Our solver successfully recovers known lower bounds (e.g.,  $k = 3$ ). However, despite exhaustive search on small instances and heuristic search on larger instances, we *fail* to find any election requiring a winning set of  $k > 3$ . Furthermore, the linear programming relaxation consistently yields values suggesting that the upper bound can be tightened to  $\alpha \leq 2/k$ . This provides strong empirical evidence that the theoretical gap lies in the upper bound, not the lower bound.

Motivated by our experimental results, we analyze the dual of the LP relaxation. We show that bounding the dual LP is sufficient to bound the primal problem. We also simplify the dual formulation significantly and conjecture that this specific structure admits a solution of at most  $2/k$ , which would imply an upper bound of four candidates in the strict majority case. This offers a new, concrete analytical path toward proving that smaller committees suffice.

**CONJECTURE 1.1 (DUAL BOUND ON  $\alpha$ ).** *Let  $u_k^*$  denote the optimal objective value of the simplified dual linear program (defined in Section 5). We conjecture that for any number of candidates  $m$  and committee size  $k$ , this value is bounded by:*

$$u_k^* \leq \frac{2}{k}$$

By weak duality, proving this conjecture would immediately imply that  $\alpha \leq \frac{2}{k}$ , meaning that Condorcet winning sets of size 4 always exist.

## 1.2 Related work

The concept of Condorcet winning sets was formalized by Elkind et al. [9], who established the initial logarithmic upper bounds and existence of size-3 lower bounds. Recent theoretical breakthroughs have focused on improving the upper bound, with Charikar et al. [8] proving existence for  $k = 6$  and Nguyen et al. [16] further tightening this to  $k = 5$ . A central open question in the field is whether the lower bound established by Charikar et al. [8] is tight. It is conjectured that for any integer  $k$ , every election admits a winning set of size  $k$  that is  $\frac{2}{k+1}$ -undominated. Proving this conjecture would close the gap between the upper and lower bounds, establishing  $\alpha = \frac{2}{k+1}$  as the limit of multi-winner representation. Our work complements these analytical approaches by using computational methods to search for counter-examples. Finding such instances would tighten the known lower bounds and disprove the conjecture; conversely, failing to find them provides empirical support for its validity.

There is a growing body of work leveraging automated reasoning to prove or disprove conjectures in social choice theory [12]. Early work successfully applied automated reasoning to rederive impossibility theorems [18, 19], such as that of Arrow [1] and Gibbard–Satterthwaite [13, 17]. Further encoding of social choice problems into SAT led to results in a number of settings [4–7, 11]. Most relevant to our work is Geist [10], who encoded the search for Condorcet winning sets as a satisfiability problem. While they successfully identified minimal elections with Condorcet dimension 3, their approach faced significant scalability hurdles. The SAT encoding requires fixing the number of candidates *and voters* in advance, leading to poor scalability.

Recently, *mixed integer linear programming (MILP)* formulations have seen success in social choice, particularly in probabilistic settings [3, 15]. Of particular relevance is the work of Berker et al. [2], who successfully applied MILP techniques to prove bounds on a property known as core stability in approval-based multi-winner voting. Our MILP formulation uses similar ideas to theirs: We relax the integrality of the voters, modeling the electorate as a probability distribution over rankings. This allows us to search the space of elections with an effectively *infinite* number of voters, removing the prior dependency. Furthermore, our enhanced MILP formulations incorporate structural symmetries to model elections with effectively infinite candidates in specific configurations (via cyclic cloning). These methodological improvements allow us to search the space of elections far more deeply than previous SAT-based attempts.

## 2 PRELIMINARIES

Prior work on Condorcet winning sets generally model an election using a set of candidates, voters, and their rankings of the candidates.

- Let  $C = \{c_1, \dots, c_m\}$  be the set of  $m$  candidates.
- Let  $N = \{v_1, \dots, v_n\}$  be the set of  $n$  voters.

- Each voter  $v \in N$  has a strict linear preference ordering  $\succ_v$  over the candidates  $C$ .

In this paper, we use the homogeneity property of CWSs (in the sense that scaling the number of votes of each kind by the same factor does not affect the results) to conclude that we do not explicitly need to model discrete voters. Let  $\mathcal{S}$  be the set of all  $m!$  possible strict linear orderings (rankings). A voter profile is a probability distribution  $x \in \Delta(\mathcal{S})$ , where  $x[s]$  is the fraction of voters with the ranking  $s \in \mathcal{S}$ . Given a ranking  $s$ , we write  $c \succ_s c'$  if candidate  $c$  is ranked higher than  $c'$  according to  $s$  and  $c \succ_s W$  if candidate  $c$  is ranked higher than each  $c' \in W$  according to  $s$ .

## 2.1 Condorcet winning set and $\alpha$ -undominated sets

We define a committee  $W \subseteq C$  that collectively dominates any candidate outside the set.

*Definition 2.1 ( $\alpha$ -undominated set).* A set of candidates  $W \subseteq C$  is an  $\alpha$ -undominated set if, for every candidate  $c \in C \setminus W$ , strictly less than an  $\alpha$  fraction of the voters prefer  $c$  over all members of  $W$ .

The fraction of voters who prefer an outside candidate  $c$  over every candidate in  $W$  according to profile  $x$  is given by:

$$\sum_{s \in \mathcal{S} | c \succ_s W} x[s]$$

Consequently,  $W$  is an  $\alpha$ -undominated set if:

$$\forall c \in C \setminus W, \quad \sum_{s \in \mathcal{S} | c \succ_s W} x[s] < \alpha.$$

*Definition 2.2 (Condorcet Winning Set).* A Condorcet winning set (CWS) is an  $\alpha$ -undominated set with  $\alpha = 1/2$ .

To illustrate the concept of a CWS, consider the election in Figure 1. This election consists of three voters  $v_1, v_2, v_3$  and three candidates  $A, B, C$ . The committee  $W = \{A, B\}$  is a CWS. The only candidate outside  $W$  is  $C$ . Only voter  $v_3$  prefers  $C$  to both  $A$  and  $B$ . Thus, the fraction of voters preferring  $C$  to  $W$  is  $1/3$ , which is less than  $1/2$ . The singleton sets are not CWSs. For example, consider  $W' = \{A\}$ . Since  $2/3$  of voters prefer  $C$  to  $A$ ,  $W'$  is not a CWS.

Voter	Ranking
$v_1$	$A \succ B \succ C$
$v_2$	$B \succ C \succ A$
$v_3$	$C \succ A \succ B$

**Figure 1: Example election consisting of voters  $v_1, v_2$  and  $v_3$  and candidates  $A, B$ , and  $C$ . The committee  $\{A, B\}$  is a CWS. The sets  $\{A\}$ ,  $\{B\}$ , and  $\{C\}$  are all not CWS.**

## 3 OPTIMIZATION FORMULATION

We are interested in finding the smallest committee size  $k$  that guarantees the existence of a CWS. Given candidates  $m$  and committee size  $k$ , we formulate this as an optimization problem maximizing the worst-case margin of defeat  $\alpha^*$  for any committee.

Let  $\mathcal{M}_k = \{W \subseteq C \mid |W| = k\}$  be the set of all committees of size  $k$ . The quantity  $\alpha^*$  represents the maximum fraction of voters who

prefer a challenger candidate  $c$  over a committee  $W$ , assuming the committee is chosen to minimize this fraction against a worst-case voter profile  $x$ .

To guarantee the existence of an  $\alpha$ -undominated set of size  $k$  over all profiles, the following condition must hold:

$$\forall x \in \Delta(\mathcal{S}), \quad \exists W \in \mathcal{M}_k, \quad \forall c \notin W: \quad \sum_{s \in \mathcal{S} | c \succ_s W} x[s] < \alpha.$$

The worst-case defeat margin  $\alpha^*$  for committees of size  $k$  is determined by the following max-min optimization:

$$\alpha^* = \max_{x \in \Delta(\mathcal{S})} \min_{W \in \mathcal{M}_k} \max_{c \notin W} \left( \sum_{s \in \mathcal{S} | c \succ_s W} x[s] \right). \quad (1)$$

We can interpret the optimization problem as a two-player zero-sum game. The *maximizer* selects a voter profile  $x$  to maximize the defeat margin  $\alpha$ , while the *minimizer* selects a committee  $W$  to minimize it. Inside the minimizer's problem, there is an implicit maximization where the strongest challenger  $c$  against  $W$  determines the value of  $\alpha$ .

The goal is to find the smallest  $k$  required to guarantee the existence of a CWS. This corresponds to finding  $k$  such that the worst-case margin  $\alpha^*$  is less than  $1/2$  for all  $m$ . Conversely, we can disprove the guaranteed existence of a CWS of size  $k$  by showing that  $\alpha^* > 1/2$  for that  $k$  and some value of  $m$ . More generally, showing that  $\alpha^* > \frac{2}{k+1}$  would disprove the open conjecture regarding the bounds of  $\alpha$ -undominated sets.

### 3.1 Basic MILP

We solve the optimization problem in Equation 1 using a mixed-integer linear program, referred to as MILP. The program takes as input the number of candidates  $m$  and the committee size  $k$ .

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & \sum_{s \in \mathcal{S}} x[s] = 1 \end{aligned} \quad (2)$$

$$\sum_{c \notin W} y_{W,c} = 1, \quad \forall W \in \mathcal{M}_k \quad (3)$$

$$\sum_{s \in \mathcal{S} | c \succ_s W} x[s] + Q(1 - y_{W,c}) \geq \alpha, \quad \forall W \in \mathcal{M}_k, c \notin W \quad (4)$$

$$x \geq 0, \quad y_{W,c} \in \{0, 1\}, \quad \alpha \geq 0 \quad (5)$$

We now describe the components of MILP in detail. The variable  $x$  is a vector indexed by the set of voter rankings  $s \in \mathcal{S}$ . Constraint 2 enforces that  $x$  is a valid probability distribution. The variable  $y_{W,c}$  is a binary variable that indicates which candidate  $c$  is the challenger to the committee  $W$ . Constraint 3 requires that each committee  $W \in \mathcal{M}_k$  has exactly one candidate challenging it. Finally, Constraint 4 enforces the max-min logic. If  $y_{W,c} = 1$ , the constraint becomes  $\alpha \leq \sum_{s \in \mathcal{S} | c \succ_s W} x[s]$ , bounding  $\alpha$  by the fraction of voters supporting challenger  $c$ . If  $y_{W,c} = 0$ , the constraint becomes  $\alpha \leq \sum_{s \in \mathcal{S} | c \succ_s W} x[s] + Q$ . By setting  $Q = 1$ , this inequality is always satisfied (as probabilities sum to at most 1) and imposes no bound on  $\alpha$ .

Since the objective is to maximize  $\alpha$ , the solver will choose a profile  $x$  and, for every committee  $W$ , identify a challenger  $c$  such that

the support for  $c$  against  $W$  is at least  $\alpha$ . This effectively computes the values of Constraint 1.

*Interpretation.* If the optimal objective value is  $\alpha^* > 0.5$ , the program has identified an election profile,  $x$ , where no Condorcet Winning Set exists. The variables  $y_{W,c} = 1$  identify the specific challengers that defeat each possible committee. In the general case, if the optimal objective value is  $\alpha^* > \frac{2}{k+1}$ , we disprove the open conjecture.

*Complexity.* We briefly mention the number of variables and constraints of MILP. The variable  $x$  is a vector of length  $|\mathcal{S}|$  which is  $O(m!)$ . Constraint 2 is a single linear constraint. The number of binary variables  $y$  is  $(m-k) \cdot \binom{m}{k}$ . Constraint 3 adds  $\binom{m}{k}$  constraints. Finally, Constraint 4 adds  $(m-k) \cdot \binom{m}{k}$  constraints, each involving a summation over the  $O(m!)$  components of  $x$ .

**3.1.1 Optimizations to the basic MILP.** The basic MILP formulation has several computational inefficiencies. We introduce four optimizations to improve the program. Note, the following optimizations are not intended to be implemented at simultaneously, but rather a variety of techniques to improve the scalability and expressivity of the MILP.

*Reducing permutations.* We can reduce the size of  $\mathcal{S}$  by observing that the specific relative ordering of candidates in the bottom  $k$  positions of a ranking does not affect the logical constraints. Consider two cases:

- (1) If a challenger  $c$  is in the bottom  $k$  positions of a ranking, and the committee  $W$  has size  $k$ , then by the Pigeonhole Principle, at least one member of  $W$  must reside in the top  $m-k$  positions. Thus,  $W$  defeats  $c$  in this ranking regardless of the specific permutation of the bottom  $k$  candidates.
- (2) If a challenger  $c$  is in the top  $m-k$  positions, it defeats any committee member located in the bottom  $k$  positions. To determine if  $c \succ_s W$ , we only need to compare  $c$  against the committee members located in the top  $m-k$  positions.

Consequently, we can group all  $m!$  permutations into equivalence classes based on their top  $m-k$  prefixes. This strictly reduces the number of  $x$  variables from  $m!$  to  $\frac{m!}{k!}$ . We always employ this reduction for MILP.

*Symmetry breaking.* The election structure contains many isomorphisms; for instance, permuting candidate labels does not change the existence of a CWS. To break this symmetry, we can fix a specific committee  $W^*$  to be the committee with the minimal max-min value and fix a specific challenger  $c^*$  to be the candidate that defeats it.

Let  $W^* = \{c_1, \dots, c_k\}$  and let the fixed challenger be  $c_{k+1}$ . We modify the program as follows:

- We omit  $W^*$  from the general constraint sets (3) and (4).
- We replace the general constraints for  $W \neq W^*$  with:

$$\sum_{c \notin W} y_{W,c} = 1 \quad \forall W \in \mathcal{M}_k \setminus \{W^*\}$$

$$\alpha \leq \sum_{s \in \mathcal{S} | c \succ_s W} x[s] + Q \cdot (1 - y_{W,c}) \quad \forall W \in \mathcal{M}_k \setminus \{W^*\}, \forall c \notin W$$

- We add a specific anchor constraint that ties  $\alpha$  directly to  $c_{k+1}$  against  $W^*$ :

$$\alpha = \sum_{s \in \mathcal{S} | c_{k+1} \succ_s W^*} x[s]$$

An alternative symmetry breaking method enforces a lexicographic ordering on the usage of challengers. Let  $L_c = \sum_{W \in \mathcal{M}_k} y_{W,c}$  be the total number of times candidate  $c$  is selected as a challenger. We can enforce:

$$L_1 \geq L_2 \geq \dots \geq L_m$$

These constraints ensure that candidates with lower indexes are used more frequently as challengers. However, unlike fixing  $W^*$ , this adds  $O(m)$  constraints that couple the binary variables across all committees.

*Subsampling permutations.* MILP requires variables for all  $|\mathcal{S}|$  permutations. However, finding a valid lower bound on  $\alpha$  does not strictly require the full set of possible rankings. A counterexample constructed from a subset of rankings is still a valid counterexample as we can set the probability on all other rankings to 0. Therefore, we can randomly sample a fixed number  $z$  of candidate permutations to form  $\mathcal{S}$ , reducing the size of  $x$  from  $O(m!)$  to  $z$ .

*Constraint generation.* The full MILP formulation contains constraints for each of the  $\binom{m}{k}$  committees. For moderate values of  $m$ , constructing the full model is intractable. However, in an optimal solution  $(x^*, \alpha^*)$ , only a small subset of committees may determine the objective value. Most committees are not preferred by strictly greater than  $\alpha^*$ .

We develop a constraint generation algorithm to avoid building the full model. Constraint generation iteratively solves a relaxed problem and adds constraints only when they are found to be violated. The algorithm consists of two components. The *Restricted Master Problem (RMP)* is a simplified version of MILP that considers only a subset of committees  $M' \subseteq \mathcal{M}_k$ . The *Separation Subproblem* uses an oracle to identify the committee  $W \in \mathcal{M}_k$  that has the lowest  $\alpha$  given the current voter distribution  $x$ . We initialize  $M'$  with a small, random subset of committees. We perform the following steps iteratively:

- (1) *Solve RMP:* We solve MILP enforcing constraints only for committees in  $M'$ . Let  $(x^{(i)}, \alpha^{(i)})$  be the optimal solution found.
- (2) *Compute separation:* We search for the committee  $W_{\text{worst}} \in \mathcal{M}_k$  against which the current distribution  $x^{(i)}$  performs the worst. We calculate the maximum support a challenger can achieve against a specific committee  $W$  as:

$$\text{score}(W, x^{(i)}) = \max_{c \notin W} \left( \sum_{s \in \mathcal{S} | c \succ_s W} x^{(i)}[s] \right).$$

The separation problem finds

$$W_{\text{worst}} = \operatorname{argmin}_{W \in \mathcal{M}_k} [\text{score}(W, x^{(i)})].$$

Let  $v_{\text{sep}} = \text{score}(W_{\text{worst}}, x^{(i)})$ .

- (3) *Update:* We compare the separation value  $v_{\text{sep}}$  with the RMP value  $\alpha^{(i)}$ .
  - If  $v_{\text{sep}} \geq \alpha^{(i)}$ , the solution is valid for all committees, and we terminate.

- Otherwise, the constraint for  $W_{\text{worst}}$  is violated. We add  $W_{\text{worst}}$  to  $M'$  and repeat.

The RMP is a relaxation of the full problem since  $M' \subseteq \mathcal{M}_k$ , so its feasible region is a superset. Since we are maximizing,  $\alpha^{(i)}$  is an upper bound on the true optimum  $\alpha^*$ :

$$\alpha^{(i)} \geq \alpha^*.$$

Conversely,  $x^{(i)}$  is a valid probability distribution for the full problem. The value  $v_{\text{sep}}$  represents the worst-case margin of this specific profile against *any* committee in the full set  $\mathcal{M}_k$ . Thus, it provides a lower bound:

$$\alpha^* \geq v_{\text{sep}}.$$

When the algorithm terminates ( $v_{\text{sep}} \geq \alpha^{(i)}$ ), we have:

$$\alpha^{(i)} \geq \alpha^* \geq v_{\text{sep}} \geq \alpha^{(i)} \implies \alpha^{(i)} = \alpha^*.$$

Since  $\mathcal{M}_k$  is finite, convergence is guaranteed. However, convergence could be considerably slower than solving MILP directly if many iterations are required.

### 3.2 Enhanced MILP

In this section, we alter MILP using a conceptual trick. The new optimization relies on an abstraction where each candidate  $c$  is treated as a representative of an infinite cycle of clones. This enhancement is necessary to produce tighter bounds, and we show experimentally that this is the case.

We assume that for any candidate  $A$ , there exists a cycle of variations  $A_1, A_2, \dots$  such that  $A_{i+1}$  defeats  $A_i$  and  $A_1$  defeats  $A_\infty$ . The cyclic assumption forces the optimization to find committees that are robust even against instances where a candidate is challenged by a variation of itself.

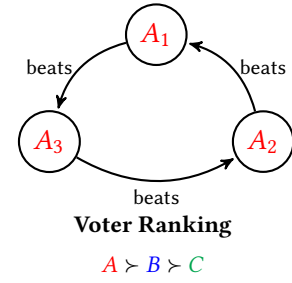
The resulting program, infMILP, produces tighter bounds and enables searching over effectively infinite candidates. Like MILP, the program takes as input the number of candidates  $m$  and the size of the committee  $k$ . However, we expand  $\mathcal{M}_k$  by considering all  $k$ -multisets of  $C$ , which we denote  $\mathcal{M}_k^+$ . In this formulation, we allow multiple copies of the same candidate to be in the same committee as well as challenge that committee.

We illustrate this concept in a finite setting in Figure 2. Candidate  $A$  is represented as a cycle of three clones,  $A_1, A_2$ , and  $A_3$ . The dominance relationships are cyclic:  $A_2$  defeats  $A_1$ ,  $A_3$  defeats  $A_2$ , and  $A_1$  defeats  $A_3$ . The entire voter population prefers  $A$  to  $B$  to  $C$ . In this election, no clone can form a CWS. As shown in the table, a committee of size 1 consisting of  $\{A_1\}$  is defeated by  $A_2$ . By symmetry, any singleton committee fails. To cover the entire cycle, the committee must contain multiple distinct clones of  $A$ . The bottom row of the table demonstrates that the multiset committee  $W = \{A, A\}$  (instantiated concretely as  $\{A_1, A_2\}$ ) is a CWS.

To formalize this, we let  $P(c, W)$  denote the fraction of voters preferring a challenger  $c$  against a multiset committee  $W$ . Let  $W(c)$  denote the multiplicity of candidate type  $c$  in  $W$ .

$$P(c, W) = \begin{cases} \sum_{s \in S | c \succ_s W} x[s] & \text{if } W(c) = 0 \\ \frac{1}{W(c)} \sum_{s \in S | c \succ_s W \setminus \{c\}} x[s] & \text{if } W(c) > 0 \end{cases}$$

**The Candidate  $A$  Cycle**



Committee	Result
$\{A_1\}$	Defeated by $A_2$
$\{A_2\}$	Defeated by $A_3$
$\{A_3\}$	Defeated by $A_1$
$\{A_1, A_2\}$	CWS; $A_1$ beats $A_3$ , $A_2$ beats $A_1$ , $A_1/A_2$ tie

**Figure 2: Illustration of the infinite cycle concept used in infMILP. The abstract candidate  $A$  is treated as a cycle of clones. While no single clone can form a CWS, the multiset committee  $W = \{A, A\}$  covers the entire cycle.**

When  $W(c) > 0$ , the challenger  $c$  is a clone of a candidate already present in the committee. The division by  $W(c)$  acts as a normalization factor. When a committee contains  $W(c)$  copies of a candidate, they can optimally distribute themselves to cover the infinite cycle, leaving at most a  $1/W(c)$  fraction of the clones preferred by any challenger. The notation  $c \succ_s W \setminus \{c\}$  denotes the orderings,  $s$ , in which  $c$  beats all candidates in the committee other than  $c$ .

Using this conceptual trick, we present infMILP below.

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & \sum_{s \in S} x[s] = 1 \end{aligned} \tag{6}$$

$$\sum_{c \in C} y_{W,c} = 1, \quad \forall W \in \mathcal{M}_k^+ \tag{7}$$

$$\sum_{s \in S | c \succ_s W} P(c, W) + Q(1 - y_{W,c}) \geq \alpha, \quad \forall W \in \mathcal{M}_k^+, c \in C \tag{8}$$

$$x \geq 0, \quad y_{W,c} \in \{0, 1\}, \quad \alpha \geq 0 \tag{9}$$

Constraints 6 and 7 ensure  $x$  is a distribution and exactly one challenger is active per committee. Constraint (8) enforces the max-min bound, using the modified definition  $P(c, W)$  to handle self-challenges via the multiset logic.

*Interpretation.* We interpret the solution of infMILP similarly to that of MILP. If the optimal objective value is  $\alpha^* > 0.5$ , the program has found an election profile where no Condorcet Winning Set exists, even when accounting for infinite clone cycles. The variables  $y_{W,c} = 1$  indicate, for each multiset committee  $W$ , which challenger

type  $c$  defeats it by at least  $\alpha^*$ . In the general case, finding  $\alpha^* > \frac{2}{k+1}$  serves as a counterexample to the open conjecture.

*Complexity.* The complexity of infMILP is strictly greater than that of MILP due to the expanded committee space. The variable  $x$  remains a vector of length  $|S|$ . The set of possible committees  $\mathcal{M}_k^+$  now includes all multisets of size  $k$ . Its size is given by the multiset coefficient  $\binom{m}{k} = \binom{m+k-1}{k}$ . The number of binary variables  $y$  is  $m \cdot |\mathcal{M}_k^+|$ . Constraint (7) contributes  $|\mathcal{M}_k^+|$  linear constraints. Constraint (8) contributes  $m \cdot |\mathcal{M}_k^+|$  linear constraints. Despite the larger constraint set, the dominant factor in the complexity remains the number of continuous variables, which is  $O(m!)$ .

### 3.3 Optimizations to the enhanced MILP

We briefly discuss how the optimizations developed for MILP also apply to infMILP.

*Reducing permutations.* We cannot reduce the size of  $S$  using the same logic as in the basic model since the Pigeonhole Principle does not apply to multisets.

*Symmetry breaking.* The election structure retains its candidate symmetries. However, we can no longer fix a specific committee  $W^* = \{c_1, \dots, c_k\}$  to be the unique minimizer, because the minimum committee might be a multiset rather than a set of distinct candidates. We can enforce the lexicographic ordering on the challengers,  $c$  in the same way as for MILP.

*Subsampling permutations.* infMILP can leverage the same subsampling logic as MILP to reduce the size of  $x$ .

*Constraint generation.* The constraint generation algorithm from MILP is directly applicable to infMILP. The only modification is that the *Separation Subproblem* now searches over the space of multisets  $\mathcal{M}_k^+$  rather than sets  $\mathcal{M}_k$  to find the committee  $W$  that maximizes the separation value.

## 4 RESULTS

We implement MILP and infMILP using the Gurobi Optimizer version 12.0.3 [14]. Experiments are conducted on a high-performance computing cluster. Each node is equipped with dual-socket AMD EPYC 7282 processors (32 physical cores, 64 logical threads) and 512 GB of RAM, running Linux kernel 4.18. Due to the cluster being shared, we restrict each trial to 64 GB of RAM.

### 4.1 Efficacy of optimizations

We first briefly evaluate the computational impact of the proposed optimizations. Table 2 summarizes the findings. For MILP, fixing the committee  $W^*$  successfully reduces the search space and improves runtime. Fixing  $W^*$  breaks symmetry without adding excessive

	Fixed $W^*$	Lexicographic challengers	Constraint generation	Subsampling
MILP	✓	✗	✗	✓
infMILP	-	✗	✗	✓

**Table 2: Effectiveness of optimizations. A ✓ indicates the method improved solve times; A ✗ indicates it was not beneficial. Note that Fixed  $W^*$  is not applicable to infMILP.**

overhead. However, this optimization is unsound for infMILP, as the optimal committee in the infinite cycle model might be a multiset rather than a set of distinct candidates. Enforcing a lexicographic ordering on challengers did not result in speedups for either model. In many cases, it slows down the solver. We hypothesize that the large number of constraints loosens the linear relaxation bounds, making the branch-and-cut process used by Gurobi less efficient. The iterative constraint generation approach also fails to produce speedups. The restricted master problem (RMP) quickly accumulates constraints until it approaches the size of the full problem, at which point the overhead of the iterative loop makes it slower than solving MILP or infMILP directly. Finally, subsampling is highly effective. By reducing the  $m!$  ranking variables to a fixed sample size  $z$ , we can scale the search to larger  $m$ . While this renders the solver incomplete, it is sufficient for finding a counterexample if one exists within the sampled profile. Note that subsampling is not sound with symmetry breaking and so we use these methods disjointly.

### 4.2 Comparison of MILP and infMILP

We compare the bounds computed by MILP and infMILP across a grid of small  $m$  and  $k$  values. Tables 3 and 4 report the maximum  $\alpha$  found. An interval  $[a, b]$  indicates the solver timed out with a feasible solution  $\alpha = a$  and a proved upper bound  $\alpha \leq b$ .

The results demonstrate the superiority of the infMILP formulation. For every instance of  $m$  and  $k$ , infMILP computes a tighter (larger) value for  $\alpha$ . The values found by MILP for  $m \leq 7$  are all below the theoretical threshold of  $\frac{2}{k+1}$  required to disprove the conjecture. This is not the case for infMILP, where we see the possibility of a counter-example in all executions that returned an upper-bound.

	$k = 2$	$k = 3$	$k = 4$
$m = 3$	0.333	-	-
$m = 4$	0.400	0.250	-
$m = 5$	0.467	0.300	0.200
$m = 6$	0.500	0.333	0.231
$m = 7$	0.524	[0.345, 0.429]	[0.250, 0.300]

**Table 3: Max  $\alpha$  values from MILP with Fixed  $W^*$ , no subsampling, and 720s timeout.**

	$k = 2$	$k = 3$	$k = 4$
$m = 3$	2/3	-	-
$m = 4$	2/3	1/2	-
$m = 5$	2/3	1/2	2/5
$m = 6$	2/3	1/2	[2/5, 1/2]
$m = 7$	2/3	[1/2, 2/3]	[2/5, 1/2]
$m = 8$	[2/3, 15/16]	-	-

**Table 4: Max  $\alpha$  values from infMILP with no subsampling and 21600s timeout.**

$m$	$k$	$z$	Iters	Timeout per Iter (s)	Counter-example	Upper-bound
7	3	4000	6	3600	✗	2/3
7	4	4000	6	3600	✗	2/4
8	2	4000	6	3600	✗	2/2
8	3	4000	6	3600	✗	2/3
8	4	4000	6	3600	✗	2/4
9	2	4000	10	3600	✗	2/2
9	3	4000	10	3600	✗	2/3
9	4	4000	10	3600	✗	2/4

**Table 5: Results of counter-example search using subsampling. A counter-example is defined as a profile where  $\alpha > \frac{2}{k+1}$ . No such examples were found.**

### 4.3 Counter-example search via subsampling

Based on the inconclusive bounds from infMILP, we target parameter settings where counter-examples could exist, such as  $m \geq 7$ . To manage the combinatorial explosion, we use subsampling with  $z = 4000$  voter rankings per iteration. Table 5 summarizes the results of over 36 hours of search. Despite extensive sampling, no counter-examples were found for  $m \in [7, 9]$ . While we attempted to search at larger  $m$ , the memory requirements for the program grew prohibitive even with subsampling.

Although no counter-example was found, a consistent pattern emerged in the solver’s upper bounds. In every run that returned an upper bound, the value was less than or equal to  $\frac{2}{k}$ . Gurobi computes upper bounds by solving LP relaxations at nodes in the branch-and-bound tree. This suggests a structural property of the formulation that naturally bounds  $\alpha$  by  $\frac{2}{k}$ .

However, this property does not hold for the root LP relaxation. Table 6 shows the values of the LP relaxation for MILP. As  $m$  increases, the relaxation value appears to approach 1. Similarly, the root relaxation for infMILP yields values  $\geq 1$  (likely due to  $Q$  loosening the bounds). The discrepancy indicates that the tighter  $\frac{2}{k}$  bound arises only after branching or cutting planes have tightened the formulation, suggesting a potential direction for a theoretical upper bound proof.

	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$m = 3$	0.333	–	–	–	–	–
$m = 4$	0.750	0.250	–	–	–	–
$m = 5$	0.886	0.667	0.200	–	–	–
$m = 6$	0.972	0.833	0.625	0.167	–	–
$m = 7$	0.999	0.901	0.777	0.600	0.143	–
$m = 8$	1.000	0.951	0.865	0.750	0.583	0.125

**Table 6: Optimal values for the LP relaxation of MILP. Note that as  $m$  increases, the relaxation appears to approach 1.**

## 5 TOWARDS AN UPPER BOUND

We present an approach for upper bounding the Condorcet dimension by analyzing the linear programming (LP) relaxation of our MILP formulation motivated by the experimental results. Since Gurobi computes upper bounds by solving LP relaxations at nodes in the branch-and-bound tree, the upper bounds of the MILP suggest that if we enforce a certain structure to the LP relaxation, we can

obtain the  $2/k$  bound. Because of this, we believe understanding the dual of the LP relaxation of MILP is an interesting avenue for proving an upper bound.

Our goal is to leverage *weak duality*: the optimal value of any dual feasible solution provides an upper bound on the primal optimal value. Ideally, if we can construct a dual solution with value  $2/k$  for any committee size  $k$ , we would theoretically prove the conjectured upper bound.

While we do not provide a closed-form proof here, we derive a significantly simplified version of the dual LP. We believe this structure captures the core difficulty of the problem and offers a concrete analytical path for future work.

### 5.1 The primal LP relaxation

First, we state the LP relaxation of our basic MILP. We relax the integrality constraints on the selection variables  $y_{W,c}$ , allowing them to take values in  $[0, 1]$ :

$$\begin{aligned}
\max \quad & \alpha \\
\text{s.t.} \quad & \sum_{s \in \mathcal{S}} x[s] = 1 \\
& \sum_{c \notin W} y_{W,c} = 1, \quad \forall W \in \mathcal{M}_k \\
& \sum_{s \in \mathcal{S} | c \succ_s W} x[s] + Q(1 - y_{W,c}) \geq \alpha, \quad \forall W \in \mathcal{M}_k, c \notin W \\
& x \geq 0, \quad y_{W,c} \in [0, 1], \quad \alpha \geq 0
\end{aligned}$$

Recall,  $Q$  is a large constant used to enforce the logical constraints. In the integer case,  $y_{W,c}$  acts as a selector. For each committee  $W$ , exactly one candidate  $c$  is active, contributing to the lower bound on  $\alpha$ .

### 5.2 Deriving the simplified dual

Taking the dual of the program above yields the following minimization problem:

$$\begin{aligned}
\min \quad & u + \sum_{W \in \mathcal{M}_k} \beta_W + Q \sum_{W \in \mathcal{M}_k, c \in C} \gamma_{W,c} \\
\text{s.t.} \quad & \sum_{W \in \mathcal{M}_k, c \in C} \gamma_{W,c} = 1 \tag{10} \\
& \beta_W + Q\gamma_{W,c} \geq 0, \quad \forall W \in \mathcal{M}_k, c \in C \tag{11} \\
& u \geq \sum_{W \in \mathcal{M}_k, c \succ_s W} \gamma_{W,c}, \quad \forall s \in \mathcal{S} \tag{12} \\
& \gamma_{W,c} \geq 0, \quad u, \beta_W \in \mathbb{R} \tag{13}
\end{aligned}$$

This formulation is complex due to the interaction between  $\beta$  and  $\gamma$ . However, we can simplify it by exploiting the properties of the parameter  $Q$ .

We observe that for sufficiently large  $Q$ , the optimal value of the primal LP is independent of  $Q$ . By strong duality, the optimal value of the dual LP must also be independent of  $Q$ . Notice that the dual objective function is linear in  $Q$ . For the optimal objective value to remain constant as  $Q$  increases, the coefficient of  $Q$  in the objective

function (after substituting optimal variable relationships) must be zero.

From constraint (11), and the fact that we are minimizing, the optimal  $\beta_W$  will be tightly bound as:

$$\beta_W = -Q \cdot \min_{c \in C} \gamma_{W,c}.$$

Substituting this into the objective function, the terms involving  $Q$  are:

$$Q \left( \sum_{W,c} \gamma_{W,c} - \sum_W \min_c \gamma_{W,c} \right)$$

For the objective value to be independent of  $Q$ , this term must vanish. Since  $\gamma_{W,c} \geq 0$ , the sum is always greater than or equal to the minimum. Equality implies that for each committee  $W$ ,  $\gamma_{W,c}$  must be concentrated on a single candidate  $c$  (or set of challengers with equal weight), effectively mimicking the integral behavior of the primal  $y_{W,c}$  variables.

### 5.3 The simplified dual conjecture

Based on this derivation, we can eliminate  $\beta_W$  and  $Q$ , resulting in the following simplified dual program:

$$\begin{aligned} \min \quad & u \\ \text{s.t.} \quad & \sum_{W \in \mathcal{M}_k, c \in C} \gamma_{W,c} = 1 \end{aligned} \quad (14)$$

$$\begin{aligned} u &\geq \sum_{W \in \mathcal{M}_k, c \succ_s W} \gamma_{W,c}, \quad \forall s \in \mathcal{S} \\ \gamma_{W,c} &\geq 0 \end{aligned} \quad (15)$$

This simplified dual has a nice interpretation. We are searching for a probability distribution  $\gamma$  over pairs  $(W, c)$ —representing a distribution of challenges—such that we minimize the maximum weight any single ranking  $s$  can cover. If we can show that for any  $k$ , there exists a distribution  $\gamma$  such that no ranking  $s$  covers more than  $2/k$  of the challenges, we will have proved the upper bound.

**CONJECTURE 5.1 (DUAL BOUND ON  $\alpha$ ).** *Let  $u_k^*$  denote the optimal objective value of the simplified dual linear program defined by constraints (14)–(15). We conjecture that for any number of candidates  $m$  and committee size  $k$ , this value is bounded by:*

$$u_k^* \leq \frac{2}{k}$$

By weak duality, proving this conjecture would immediately imply that  $\alpha \leq \frac{2}{k}$ , meaning that Condorcet winning sets of size 4 always exist. We can reformulate the conjecture to equivalently state the following.

**CONJECTURE 5.2 (DUAL BOUND PROBABILISTIC INTERPRETATION).** *There exists a probability distribution  $\gamma$  over the set of committee-challenger pairs  $\{(W, c) \mid W \in \mathcal{M}_k, c \in C \setminus W\}$  such that for every possible ranking  $s \in \mathcal{S}$ , the total probability mass of pairs where the challenger defeats the committee is at most  $2/k$ :*

$$\max_{s \in \mathcal{S}} \sum_{W \in \mathcal{M}_k} \sum_{\substack{c \in C \setminus W \\ c \succ_s W}} \gamma_{W,c} \leq \frac{2}{k}$$

where  $c \succ_s W$  denotes that candidate  $c$  is preferred to every member of committee  $W$  in ranking  $s$ .

## 6 CONCLUSION AND FUTURE WORK

In this work, we introduced a *mixed-integer linear programming (MILP)* framework to search for elections with high Condorcet dimensions. By modeling the electorate as a continuous probability distribution and exploiting structural symmetries, our approach scales significantly better than previous SAT-based attempts, allowing us to search over effectively infinite candidates.

While our solver successfully recovered the known lower bound constructions (*i.e.*,  $k = 3$ ), its inability to find harder instances despite extensive search provides compelling empirical evidence that the current theoretical lower bounds are likely tight. Specifically, our experimental results consistently respect the bound  $\alpha \leq 2/k$ . If this bound holds generally, it implies that a committee of size  $k = 4$  is sufficient to win a voter majority—tightening the best known upper bound of  $k = 5$ .

We have taken the first step toward proving this conjecture by analyzing the dual of the LP relaxation. We demonstrated that the dual structure can be significantly simplified, offering a concrete analytical path to improving the upper bounds.

Despite the effectiveness of our MILP formulation, our results remain empirical. While the upper bounds computed during search provide evidence for the  $2/k$  conjecture, it does not constitute a formal proof. Our approach offers many advantages over SAT-based methods, such as abstracting away the number of voters. However, this abstraction of voters causes the search space of candidate permutations to grow exponentially with  $m$ . Consequently, for large numbers of candidates, our solver relies on heuristics and time-bounded search rather than exhaustive verification, leaving open the possibility that worst-case structures exist beyond our computational horizon.

For future work, the most promising direction is the formal analysis of the simplified dual linear program. Proving that the optimal solution to the dual is bounded by  $2/k$  would begin to close the gap between the existence and impossibility results in this domain. Additionally, the automated reasoning framework, including the optimizations we make, can be extended to other open problems in social choice, opening new avenues to tackle open conjectures.

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