Fair Allocation of Improvements: When Old Endowments Shape New Assignments

Noga Klein Elmalem
The Open University of Israel
Raanana, Israel
noga486@gmail.com

Rica Gonen
The Open University of Israel
Raanana, Israel
ricagonen@gmail.com

Erel Segal-Halevi
Ariel University
Ariel, Israel
erelsgl@gmail.com

ABSTRACT

Background: This work is motivated by the urban renewal process called Reconstruct and Divide, where old buildings are demolished and homeowners receive upgraded apartments. Although surplus units make the project potentially win–win, many projects are delayed due to disputes over the fairness of the new assignment. We aim to design algorithms for fair assignment of the new apartments with monetary transfers.

Objectives and Research Questions: Unlike classical models, envy depends on the value of the old apartments, as owners of more valuable units expect greater improvements. We study (1) how to allocate apartments and payments when envy-free or proportional outcomes may not exist, and (2) how to elicit truthful valuations despite incentives to inflate old-apartment values.

Methods: We model utilities as improvements over previous apartments and focus on minimizing maximal envy and disproportionality. We develop graph-based and optimization algorithms for computing payments and assignments, and introduce a characteristic-based elicitation method to mitigate strategic manipulation.

Results: We present a strongly polynomial-time algorithm that, for a given assignment, computes payments minimizing maximum pairwise envy, and another algorithm that finds an assignment and payments minimizing maximum disproportionality. We further identify conditions under which the Minimum Disproportionality mechanism is risk-averse truthful.

Conclusions: Fair division with endowments requires new computational and strategic tools beyond standard envy-free models. Our algorithms and elicitation approach provide practical foundations for Reconstruct and Divide projects while exposing open challenges in minimizing envy over assignments.

KEYWORDS

Fair assignment; House allocation; Real estate; Endowments; Novel application; GTEP

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1 INTRODUCTION

Urban renewal aims to transform aging residential infrastructure into modern, safe, and more spacious housing while simultaneously increasing urban density.

Reconstruct and Divide is a common urban renewal process. It involves demolishing an old building and constructing a new one, with original homeowners receiving upgraded apartments as compensation. While the primary goals include enhancing urban housing availability and improving disaster resilience, the success of these projects is often impeded by disputes among stakeholders, particularly concerning the assignment of new apartments. These disagreements commonly stem from perceptions of unfairness, as homeowners compare the value of their newly-assigned units to others. Such cases often get to court and lead to lengthy judicial processes, leading to delays or even project cancellations.

Example 1.1. In one court proceeding we found the following claims by one homeowner: (1) Her new apartment is larger than her old apartment by 19 sqm, whereas the new apartments of other homeowners are larger by 23 sqm; (2) her new apartment is 1.5 floors higher than her old apartment, whereas the new apartments of other homeowners are higher by 2 floors; (3) her old apartment was in a half-floor with no adjacent neighbors, whereas her new apartment has adjacent neighbors; (4) her old apartment was square-shaped whereas her new apartment is not, but other homeowners did get a square-shaped apartment.

These claims were rejected by the judges, as they claimed that fairness only requires the equal treatment of equals, whereas homeowners with different old apartments are not apriori equal. However, such subjective feelings of envy might lead to homeowners disagreeing to enter the process in the first place, foregoing its advantages for all parties involved.

The baseline for this research is the problem often dubbed *Rental Harmony* [10], where some n agents should be assigned some n items (rooms or apartments) and some monetary payments, such that no agent envies the bundle (item + payment) of another agent. An envy-free assignment is known to exist under very broad conditions [2, 4, 16, 23–26, 28, 32, 33], and existing computational solutions are widely applied in practice [12].

The main challenge in the Reconstruct and Divide setting is that, as illustrated by Example 1.1, the envy of agents is determined not by the value of their new apartment alone, but by the *improvement* of their new apartment over their old one. Even before going into details of how this improvement is computed, we can already see that an envy-free assignment might not exist.

Example 1.2. Suppose each agent believes that his old apartment is more valuable than all other old apartment, but the new apartments are all identical. In this case, regardless of the assignment, each agent will feel that the other agents got a better improvement. Payments might reduce the envy of some agents but increase the envy of others; no payment vector can totally eliminate the subjective envy (See Example 3.3 for a formal proof).

A common solution to this issue is to hire an appraiser to compute the market value of old apartments. However, as illustrated by Example 1.1, agents' valuations are subjective and may differ from the market values. A central challenge in this work is to identify and formalize fairness and strategic guarantees that are both meaningful and attainable in the Reconstruct and Divide setting.

As it might be impossible to completely eliminate envy, we focus on minimizing the envy of the most envious agent (see Sections 3 and 4 for the formal definition).

Proportionality is another prominent fairness notion. In the standard fair division setting (with no old allocation), proportionality means that each agent should receive a bundle worth (subjectively) at least $1/n$ of the total, where the "total" is simply the sum of (subjective) values of all items. In our setting, as the allocation is of improvements, it makes sense to define the "total" as the sum of (subjective) improvements of all items. For example, if you value all old apartments together at 100 and all new apartments together at 150, then proportionality guarantees that your own subjective value will improve by at least $50/n$.

Proportionality is weaker than envy-freeness: every envy-free allocation is proportional, but not the other way around. This holds both in the standard setting and in our setting (see Section 4). Still, a proportional allocation might not exist. This can be seen in Example 1.2: each agent believes that his improvement is smaller than the improvement of any other agent; hence, his improvement is necessarily smaller than $1/n$ the sum of improvements (see Section 4 for a formal proof). Similarly to the case of envy-freeness, we aim to minimize the maximum "dis-proportionality" — the difference between each agent's proportional share to the agent's improvement.

1.1 Our Results

We assume throughout the paper that (a) all agents have quasilinear utilities; (b) all agents evaluate their improvements by computing the *difference* between the values of their new apartments and their old apartments[1].

In Section 3 we study envy-freeness (EF). We provide a necessary and sufficient condition for an assignment to be "EF-able" (can be made EF using payments). Since an EF-able assignment might not exist, we aim to identify, for any given assignment, a payment-vector that minimizes the maximum envy over all agents. We prove that this minimum equals the *maximum average cycle cost* in the envy-graph corresponding to the assignment. Hence, we can use existing algorithms to minimize the maximum envy for a given assignment in strongly-polynomial time[2]. However, we do not yet have a polynomial-time algorithm for finding an assignment that minimizes the maximum envy. Whether such an algorithm exists or not remains an intriguing open problem.

In Section 4 we study proportionality (PROP). We provide a necessary and sufficient condition for an assignment to be PROP-able. We show that it is possible to find, in polynomial time, an assignment and a payment-vector that minimize the maximum dis-proportionality over all agents; we call it the *Minimum Disproportionality* mechanism. Notably, this mechanism is based on a simple utilitarian-maximization algorithm originally developed for computing envy-freeable allocations. We show that it can be repurposed to minimize disproportionality in the more complex setting with endowments.

As the PROP model is more computationally tractable, we focus on this model and discuss, in Section 5, its strategic aspects. It is well-known that, even in the basic Rental Harmony setting, no fair and budget-balanced algorithm is truthful. But in the Reconstruct and Divide setting the situation is much worse, as the agents have a strong incentive to report the maximum possible value to their old apartment. In fact, this manipulation is a dominant strategy — it is a *safe manipulation* (see Section 5 for definitions). To alleviate this issue, we introduce a new way to elicit agents' valuations: instead of asking them to directly report the values of old and new apartments, we ask them to report the values of apartments' *characteristics*, such as apartment orientation, high floor etc. The reported characteristic may appear both in old and in new apartments. We explore the conditions under which the Minimum Disproportionality mechanism has no safe manipulations. To the best of our knowledge, characteristic-based elicitation has not previously been used to improve strategic properties of fair-division mechanisms.

In [11], we simulated a Reconstruct and Divide scenario using survey-based valuations from 45 homeowners, comparing outcomes under minimizing envy and minimizing disproportionality models. The results reveal the gap between the models and underscore the importance and the need for improved allocations in the minimizing envy model. Our experiments also provide a novel dataset of valuations over item characteristics and show that the Minimum Disproportionality mechanism often attains better-than-proportional outcomes.

All omitted proofs appear in the full version [11].

1.2 Related Work

Our work connects several lines of research, including fair assignment with monetary transfers, initial endowments, variations of the house assignment problem, and relaxations of truthfulness.

The problem of minimizing subsidies to achieve envy-freeness was introduced by Halpern and Shah [17], who showed that any assignment can be made envy-free using monetary transfers. Brustle et al. [7] extended this idea by allowing the assignment itself to vary and by providing tight bounds on the required subsidies. More recently, Kawase et al. [21] improved the known subsidy bounds by showing that from any EF1 allocation one can compute in polynomial time an envy-free allocation with per-agent subsidy at most

[1] We considered an alternative model in which agents compute *ratios* rather than differences, but could not yet get any interesting results for this model. See [11] for details.

[2] Strong polynomial time is a well-known concept in computer science, applicable to problems with numbers. Whereas a polynomial-time algorithm depends on the bit-length of input numbers, a strongly polynomial-time algorithm depends only on the number of input numbers. Strongly polynomial time is better as it avoids slowdowns from large or precise numbers.

$n-1$ and total subsidy at most $n(n-1)/2$, with even better bounds for monotone valuations.

Goko et al. [14] study an algorithm for computing envy-free allocations with subsidies that is also *truthful*, when agents have submodular binary valuations.

Wu et al. [37, 38] study proportional allocations with subsidies. They show that under additive valuations a total subsidy of $n/4$ is sufficient and tight to guarantee proportionality, and provide improved bounds and rounding schemes for the weighted case.

Finally, Dai et al. [8] investigate weighted envy-freeness in the house allocation setting, presenting polynomial-time algorithms and structural characterizations of weighted envy-free and weighted envy-freeable assignments, under a definition of weighted envy-freeness that differs from the one adopted in our work.

For a broader perspective, Liu et al. [22] survey recent advances in fair division and highlight open problems in mixed settings involving goods, chores, divisible and indivisible resources, and subsidies.

In the specific context of house assignment—where each agent receives exactly one item—few works address envy-freeness. Gan et al. [13] proposed a polynomial-time algorithm to decide and compute envy-free assignments in such settings.

Our setting also draws on the literature addressing repeated allocations, where items are assigned multiple times rather than just once. Two main approaches have been considered: one, as proposed in our work, aims to preserve relative envy across repetitions; the other focuses on balancing envy over time. Balan et al. [5] study long-term fairness by prioritizing the most disadvantaged agents in repeated decisions. Igarashi et al. [19] introduce a model for repeated allocation of goods and chores, proving the existence of proportional and Pareto-optimal sequences under certain conditions, while noting that envy-freeness may require relaxed criteria or flexible repetition counts. To our knowledge, repeated fair allocation with monetary transfers has not yet been addressed.

Another relevant body of work concerns fair assignments with initial endowments and strategic considerations. Yılmaz [40] introduced the Probabilistic Serial mechanism to incorporate private endowments, ensuring ordinal efficiency and individual rationality via a generalized eating algorithm [6], though at the cost of strategy-proofness. More recently, Thomson [29, 30] demonstrated that most standard mechanisms are manipulable, even under assumptions such as homothetic preferences or penalties for dishonesty. In the housing context, Abdulkadiroğlu and Sönmez [1] designed a mechanism that is strategy-proof, Pareto-efficient, and respects existing tenants' rights. Finally, Segal-Halevi [27] addressed re-division problems, proposing mechanisms that balance fairness with respect for ownership, and analyzing trade-offs using the price-of-fairness metric.

See [11] for further related work.

2 PRELIMINARIES

Agents and valuations. We denote by $[t]$ the set $\{1, 2, ..., t\}$ for any positive integer t. There are n agents; the set of agents is denoted $N = [n]$. The set of old apartments is denoted $O = \{o_1, \dots, o_n\}$, where o_i is the old apartment owned by agent i. The set of new apartments is denoted $M = \{a_1, ..., a_n\}$.

An *assignment* is a bijective mapping from agents to new apartments. It is denoted $A = (A_1, \dots, A_n)$, where $\forall i \in N : A_i \in M$, and $A_i \neq A_j$ for all $i \neq j \in N$, and A_i is the new apartment assigned to agent i.

Each agent $i \in N$ has a *valuation function* $v_i : M \cup O \to \mathbb{R}_{\geq 0}$, indicating how much they value different apartments. So $v_i(A_i)$ represents agent i's valuation of their new apartment under assignment A. Note that the classic model is equivalent to assuming that all agents value all old apartments at 0.

In addition to receiving item A_i, each agent $i \in N$ is given a payment p_i, where the vector of payments is denoted as $\mathbf{p} = (p_1, ..., p_n)$. These payments can be either positive or negative. A payment vector $\mathbf{p}$ is said to be *balanced* if $\sum_{i \in N} p_i = 0$.

We assume that agents are *quasilinear*, so that the utility of each agent for an apartment and payment is

$$u_i(A_i, p_i) := v_i(A_i) + p_i.$$

Moreover, we assume that the subjective happiness of each agent is determined by how much his new bundle (new apartment plus payment) is better than his old apartment. We measure this improvement by the *difference*:[1]

$$d_i(a, o, p) := u_i(a, p) - v_i(o) = v_i(a) - v_i(o) + p.$$

So the subjective happiness of agent i given allocation A and payment vector $\mathbf{p}$ is $d_i(A_i, o_i, p_i)$; and the subjective happiness that i attributes to some other agent j is $d_i(A_j, o_j, p_j)$. We also denote $d_i(a, o) := d_i(a, o, 0)$.

Utilitarian social welfare. The *utilitarian social welfare* of an assignment A is $\sum_{i \in N} v_i(A_i)$. An assignment A of the new apartments is called *utilitarian welfare maximizing* if $\sum_{i \in N} v_i(A_i) \geq \sum_{i \in N} v_i(B_i)$ for any other assignment B.

3 MINIMIZING ENVY

Given an assignment A and payment-vector $\mathbf{p}$, we define the *envy* felt by an agent i by:

$$ENVY_i(A, \mathbf{p}) := \max_{j \in N} d_i(A_j, o_j, p_j) - d_i(A_i, o_i, p_i).$$

The pair $(A, \mathbf{p})$ is called *Envy Free (EF)* if $ENVY_i(A, \mathbf{p}) \leq 0$ for all $i \in N$. Equivalently, $d_i(A_i, o_i, p_i) \geq d_i(A_j, o_j, p_j)$ for all $i, j \in N$. An assignment A is called *EF-able* if there exists a payment vector $\mathbf{p}$ such that $(A, \mathbf{p})$ is EF. Note that this definition reduces to the standard definition of envy-freeness when there are no old apartments (equivalently, when $v_i(o_j) = 0$ for all $i, j \in N$).

3.1 The Envy Graph

To analyze the conditions for EF-ability, we extend the notion of *envy-graph*, which was introduced by Aragones [3] and studied also by Halpern and Shah [17].

The *envy-graph* of an assignment A and a payment vector $\mathbf{p}$, denoted $G_{A,\mathbf{p}}$, is a complete directed graph in which the set of nodes is the set N of agents. Each pair of agents is connected by arcs in both directions. For any pair of agents $i, j \in N$, the arc (i, j) has a *cost* defined as the envy i feels toward j under assignment A, that is, $cost_A(i, j) := \big(v_i(A_j) + p_j\big) - \big(v_i(A_i) + p_i\big)$. We denote the cost of a path $(i_1, ..., i_k)$ as $cost_A(i_1, ..., i_k) = \sum_{j=1}^{k-1} cost_A(i_j, i_{j+1})$.

For convenience, when the payment vector is zero (i.e., no payments are made), we denote it by G_A.

In addition to the envy-graph G_A of the new apartments assignment, we are also interested in the following envy-graphs:

- **The original apartments graph,** G_O, where the cost of every arc (i,j) is $cost_O(i,j) := v_i(o_j) - v_i(o_i)$;
- **The difference graph,** $G_{A,O}$, where the cost of every arc (i,j) is $(v_i(A_j) - v_i(o_j)) - (v_i(A_i) - v_i(o_i)) = cost_A(i,j) - cost_O(i,j)$. If the weights on each graph are represented by a matrix, then the matrix of $G_{A,O}$ is simply the difference between the matrix of G_A and the matrix of G_O; shortly, $G_{A,O} = G_A - G_O$.

Using the difference envy-graph, we can extend the characterization of Halpern and Shah [17] to the Reconstruct and Divide setting.

Theorem 3.1. *For O the assignment of the original apartments and A the new apartments assignment, the following are equivalent:*

(a) A is EF-able;
(b) $\sum_{i\in N} v_i(A_i) - v_i(o_i) \geq \sum_{i\in N} v_i(A_{\pi(i)}) - v_i(o_{\pi(i)})$, for all permutations π of N;
(c) The difference graph $G_{A,O}$ has no positive-cost cycles; equivalently, for every directed cycle C, $cost_A(C) \leq cost_O(C)$.

Proof. $(a) \Rightarrow (b)$: Suppose A is EF-able. Then, there exists a payment vector $\mathbf{p}$ such that$(A,\mathbf{p})$ is EF. that is, for all agents i,j, $v_i(A_i) - v_i(o_i) + p_i \geq v_i(A_j) - v_i(o_j) + p_j$. Equivalently,

$$\big(v_i(A_j) - v_i(o_j)\big) - \big(v_i(A_i) - v_i(o_i)\big) \leq p_i - p_j.$$

Consider any permutation π of N. Then,

$$\sum_{i\in N}\Big(\big(v_i(A_{\pi(i)}) - v_i(o_{\pi(i)})\big) - \big(v_i(A_i) - v_i(o_i)\big)\Big) \leq \sum_{i\in N}\Big(p_i - p_{\pi(i)}\Big) = 0.$$

The last entry is zero as all the payments are considered twice, and they cancel out each other. Hence,

$$\sum_{i\in N} v_i(A_i) - v_i(o_i) \geq \sum_{i\in N} v_i(A_{\pi(i)}) - v_i(o_{\pi(i)})$$

for each permutation π of N.

$(b) \Rightarrow (c)$: Suppose that $\sum_{i\in N} v_i(A_i) - v_i(o_i) \geq \sum_{i\in N} v_i(A_{\pi(i)}) - v_i(o_{\pi(i)})$, for some assignment A, and all permutations π of N. Consider a cycle $C = (i_1, ..., i_r)$ in $G_{A,O}$, and a permutation π, defined for each agent $i_k \in N$ as follows:

$$\pi(i_k) = \begin{cases} i_k, & i_k \notin C \\ i_{k+1}, & k \in \{1,\dots,r-1\} \\ i_1 & k = r \end{cases}.$$

Examining the cost of C in the difference graph $G_{A,O}$:

$$cost_A(C) - cost_O(C) = cost_{A,O}(C) = \sum_{i\in C}\Big(\big(v_i(A_{\pi(i)}) - v_i(o_{\pi(i)})\big) - \big(v_i(A_i) - v_i(o_i)\big)\Big) \leq 0.$$

The validity of the equality arises from the assumption.

$(c) \Rightarrow (a)$: Suppose $G_{A,O}$ has no positive-cost cycles. Then, we can define for each agent i, the maximum-cost of any path in the difference graph that starts at i. We denote the cost of this path by ℓ_i. For each $i \in N$, let $p_i = \ell_i - \frac{\sum_{i\in N}\ell_i}{n}$. It is noteworthy that $\sum_{i\in N} p_i = 0$, establishing the suitability of $\mathbf{p}$ as a balanced payment vector. Furthermore, in accordance with the definition of highest-cost paths, it follows that for all $i \neq j \in N$:

$$p_i = \ell_i - \frac{\sum_{i\in N}\ell_i}{n} \geq cost_{A,O}(i,j) + \ell_j - \frac{\sum_{i\in N}\ell_i}{n} =$$
$$\big(v_i(A_j) - v_i(o_j)\big) - \big(v_i(A_i) - v_i(o_i)\big) + \ell_j - \frac{\sum_{i\in N}\ell_i}{n} =$$
$$v_i(A_j) - v_i(o_j) + p_j - \big(v_i(A_i) - v_i(o_i)\big) \iff$$
$$v_i(A_i) - v_i(o_i) + p_i \geq v_i(A_j) - v_i(o_j) + p_j.$$

Hence, $(A,\mathbf{p})$ are envy-free, and thus, A is EF-able. □

With no old apartments, any utilitarian-welfare-maximizing assignment welfare is EF-able [16, 28]. We extend this as follows.

Proposition 3.2. *Let A be a utilitarian-welfare-maximizing assignment. Then A is EF-able if one of the following holds:*

(a) The original envy-graph G_O has no negative-cost cycles;

(b) All n agents assign the same values (e.g. market prices) to the original apartments.

Proof. (a) If A is utilitarian-welfare-maximizing, then G_A has no positive-cost cycles [17]. For each directed cycle C, its cost in $G_{A,O}$ equals its cost in G_A minus its cost in G_O. As G_O has no negative-cost cycles, the cost of C in $G_{A,O}$ is non-positive. Hence A is EF-able by Theorem 3.1.

(b) Under the given conditions, the cost of every directed cycle in O is zero. Hence A is EF-able by part (a). □

On the negative side, there are cases in which no assignment of the new apartments is EF-able. The following example formalizes the intuitive explanation given in Example 1.2.

Example 3.3. Suppose there are two agents i_1 and i_2 with valuations:

$$\begin{bmatrix} & o_1 & o_2 & a_1 & a_2 \\ i_1 & z+\epsilon & z & V_1 & V_1 \\ i_2 & y & y+\delta & V_2 & V_2 \end{bmatrix}.$$

when $y, z, \epsilon, V_1, V_2 > 0$. Let C be the cycle $i_1 \to i_2 \to i_1$. Then $cost_O(C) = -(\epsilon + \delta)$. In contrast, for any assignment A of the new apartments, $cost_A(C) = 0$. Hence,

$$cost_{A,O}(C) = cost_A(C) - cost_O(C) = \epsilon + \delta > 0.$$

By Theorem 3.1, A is not EF-able. □

3.2 Max average cycle cost and min envy

Since EF-able assignments may not exist (Example 3.3), our goal shifts to minimizing the largest envy experienced by an agent. We begin with a lemma demonstrating that envy may remain despite efforts to eliminate it using balanced payments.

Lemma 3.4. *The total cost of any cycle in $G_{A,O}$ is the same for any payment vector.*

Proof. Suppose we give some agent i a payment of p_i. As a result, the cost of every edge from i decreases by p_i (as i experiences less envy), and the cost of every edge into i increases by p_i (as other agents experience more envy in i). Every cycle through i

contains exactly one edge from i and one edge into i, and every other cycle contains no such edges. Therefore, the total cost of any cycle remains unchanged. □

Now, we demonstrate how to construct a payment vector for a given assignment that minimizes the largest envy, that is, we solve the problem $\min_{\mathbf{p}} \max_{i \in N} ENVY_i(A, \mathbf{p})$. Although this can be done by solving a linear program, our algorithm runs in strongly-polynomial time and also reveals interesting links between our problem and fundamental graph-theoretic concepts.

Definition 3.5. Given a directed graph G with edge costs, The *average cost* of a path is the total cost of the path divided by the number of edges in it. The *maximum average cycle cost (MACC)* of G is the maximum, over all directed cycles C in G, of the average cost of C.

In the literature, the MACC of G is also known as its *maximum mean cycle weight* [20]. The MACC of any given directed graph can be computed in strongly-polynomial time ([20, 31]). The following lemma relates the MACC to the minimum attainable envy.

Lemma 3.6. *For a given assignment A, it is possible to compute in polynomial time a payment vector* $\mathbf{p}$ *such that the envy between any two agents is bounded by the maximum average cycle cost of $G_{A,O}$. Specifically, for all $i, j \in N$, the following holds:*

$$d_i(A_j, o_j, p_j) - d_i(A_i, o_i, p_i) \leq c,$$

where $c := MACC(G_{A,O})$.

Proof. Let C be a cycle with the maximum average cost in $G_{A,O}$. meaning that for any other cycle C', it holds that $c = \frac{cost_A(C)}{|C|} \geq \frac{cost_A(C')}{|C'|}$.

Temporarily modify $G_{A,O}$ to obtain $G'_{A,O}$, where the cost of each edge is its original cost in $G_{A,O}$ minus c. The cost of every cycle C' in $G'_{A,O}$ is thus its cost in $G_{A,O}$ minus $c|C'|$, which is greater than $\frac{cost_A(C')}{|C'|}|C'| = cost_A(C')$, meaning there are no positive-cost cycles:

$$cost_A(C') - c|C'| \leq cost_A(C') - \frac{cost_A(C')}{|C'|}|C'| =$$
$$cost_A(C') - cost_A(C') = 0.$$

By Theorem 3.1, there exists a payment vector $\mathbf{p}$ that eliminates all envy in $G'_{A,O}$. Specifically, for each agent $i \in N$, define

$$p_i = \ell_i - \frac{\sum_{i \in N} \ell_i}{n},$$

where ℓ_i denotes the cost of the maximum-cost path in $G'_{A,O}$ originating from i. Moreover, such a vector can be computed in polynomial time: initially, the Floyd-Marshall algorithm (Weisstein [34], Wimmer and Lammich [36]) is applied to the graph derived by negating all edge costs in $G'_{A,O}$ (This has a linear time solution since there are no cycles with positive costs in the graph). Hence, determining the longest path cost between any two agents, accomplished in $O(nm + n^3)$ time. Subsequently, the longest path starting at each agent is identified in $O(n^2)$ time. Using this payment vector, we have for each $i, j \in N$:

$$d_i(A_j, o_j, p_j) \leq d_i(A_i, o_i, p_i) + c \iff$$
$$d_i(A_j, o_j, p_j) - d_i(A_i, o_i, p_i) \leq c.$$

Hence, $ENVY_i(A, \mathbf{p}) \leq c$ for all $i \in N$. □

Lemma 3.7. *Let A be an assignment of the new apartments. Then*

$$\min_{\mathbf{p}} \max_{i \in N} ENVY_i(A, \mathbf{p}) = MACC(G_{A,O}),$$

and a minimizing $\mathbf{p}$ *can be computed in polynomial time.*

Proof. Let C be a maximum average-cost cycle in $G_{A,O}$, and let c be its average cost. By Lemma 3.4, the cost of a cycle remains the same under any payment vector $\mathbf{p}$, and the same holds for the average cost. By the pigeonhole principle, the cycle has at least one edge with cost at least c; this means that, for any payment vector, the maximum envy in A is at least c. This proves the $\geq$ direction.

By Lemma 3.6, a payment vector for which the envy of all agents is at most c exists and can be computed in polynomial time; this proves the $\leq$ direction. □

To illustrate Lemma 3.7, consider Example 3.3 again. There is only one directed cycle, namely $i_1 \to i_2 \to i_1$; its cost in $G_{A,O}$ is $\epsilon+\delta$, so its average cost is $(\epsilon+\delta)/2$. Indeed, with the payment vector $p_1 = (-\epsilon + \delta)/2, p_2 = (\epsilon - \delta)/2$, the envy of agent 1 modifies from ϵ to $(\epsilon + \delta)/2$ and the envy of agent 2 modifies from δ to $(\epsilon + \delta)/2$, so both agents attain the envy bound. With every other payment vector, one agent would experience envy higher than $(\epsilon + \delta)/2$.

As noted earlier, the MACC of any directed graph can be computed in strongly polynomial time. Hence, by Lemma 3.7, we can find $\mathbf{p}$ minimizing $\max_{i \in N} ENVY_i(A, \mathbf{p})$ in strongly-polynomial time.

However, minimizing the largest envy over all assignments is much more challenging, as by Lemma 3.7, it requires to find an assignment A that minimizes $MACC(G_{A,O})$.

Open Question 3.8. Is there a polynomial-time algorithm that, given an assignment O of the original apartments, computes an assignment A of the new apartments that minimizes $MACC(G_{A,O})$?

4 MINIMIZING DISPROPORTIONALITY

Given an assignment A and a payment vector $\mathbf{p}$, we define the *total improvement* for agent i as

$$TOTAL_i := \sum_{j \in N} d_i(A_j, o_j, p_j).$$

As p is a balanced payment vector the payments cancel out, so an equivalent definition is $TOTAL_i = \sum_j (v_i(A_j) - v_i(o_j))$, which is simply the sum of all new apartments' values minus the sum of all old apartments' values in i's eyes; hence, $TOTAL_i$ does not depend on $\mathbf{p}$ nor on A.

An allocation $(A, \mathbf{p})$ is called *proportional* if each agent enjoys at least a fraction $1/n$ of the total improvement, that is, $d_i(A_i, o_i, p_i) \geq \frac{1}{n} TOTAL_i$ for all $i \in N$.

An assignment A is called PROP-able if there exists a payment vector $\mathbf{p}$ such that $(A, \mathbf{p})$ is proportional.

It is well-known in other fair division domains that envy-freeness implies proportionality, and when $n = 2$ the opposite implication holds too. The same implications exist in our domain.

PROPOSITION 4.1. *(a) If an allocation A and a payment vector $\mathbf{p}$ are envy-free, then they are also proportional.*

(b) When $n = 2$, if $(A, \mathbf{p})$ is proportional, then it is also envy-free.

PROOF. (a) If $(A, \mathbf{p})$ is envy-free, then by definition, for all $i \in N$,

$$d_i(A_i, o_i, p_i) = \max_{j \in N} d_i(A_j, o_j, p_j).$$

As the maximum is always at least as high as the average, this implies $d_i(A_i, o_i, p_i) \geq TOTAL_i/n$, so $(A, \mathbf{p})$ is proportional.

(b) When there are only two agents, if $d_i(A_i, o_i, p_i)$ is at least as large as the average, then it must be maximum, so $(A, \mathbf{p})$ is EF. □

A PROP-able allocation may still not exist. This is shown by the same Example 3.3, as in this example there are two agents. Hence, as in the previous section, we aim to minimize the largest *deviation* from proportionality, which we define by $DP_i(A, \mathbf{p}) := \frac{1}{n} TOTAL_i - d_i(A_i, o_i, p_i)$.

We start by minimizing the largest disproportionality for a given assignment, that is, solving $\min_{\mathbf{p}} \max_{i \in N} DP_i(A, \mathbf{p})$.

We denote by $DP_i(A)$ the disproportionality of i when the assignment is A and all payments are 0. Note that

$$DP_i(A) = \frac{1}{n} [\sum_{j \neq i} d_i(A_j, o_j) - (n-1) d_i(A_i, o_i)].$$

We denote the *total disproportionality* of assignment A by

$$DP_N(A) := \sum_{i \in N} DP_i(A).$$

LEMMA 4.2. *Let A be an assignment of the new apartments. Then*

$$\min_{\mathbf{p}} \max_{i \in N} DP_i(A, \mathbf{p}) = \frac{DP_N(A)}{n},$$

and the minimizing $\mathbf{p}$ is given by

$$p_i = DP_i(A) - \frac{DP_N(A)}{n} \tag{1}$$

Note that the payment vector given by (1) is balanced.

Lemma 4.2 implies that, in order to minimize the largest disproportionality over all assignments, one should find an assignment A that minimizes $DP_N(A)$. We show below that this minimum is attained by any utilitarian-welfare-maximizing assignment.

LEMMA 4.3. *If an assignment A maximizes the utilitarian social welfare, then it minimizes $DP_N(A)$ over all assignments.*

Hence, A minimizes $\min_{\mathbf{p}} \max_{i \in N} DP_i(A, \mathbf{p})$.

Consequently, we propose a polynomial-time mechanism for Reconstruct and Divide projects that computes an assignment and payment vector minimizing the maximum disproportionality among all agents:

Algorithm 1 Minimum Disproportionality Mechanism

1: Find an assignment A maximizing the utilitarian welfare;
2: Compute a payment vector $\mathbf{p}(A)$ by Equation (1).

As a corollary, we get that the existence of proportional allocations can be decided in polynomial time.

COROLLARY 4.4. *For O, the original apartments, and M, the new apartments, an assignment A and a balanced payment vector $\mathbf{p}(A)$ such that $(A, \mathbf{p}(A))$ is proportional can be found, or it can be determined that no such assignment and vector exist, all in polynomial time.*

PROOF. Use Algorithm 1 to compute $(A, \mathbf{p})$. Then compute

$$\max_{i \in N} DP_i(A, \mathbf{p}).$$

If $\max_{i \in N} DP_i(A, \mathbf{p}) \leq 0$, then $(A, \mathbf{p})$ is proportional by definition. Otherwise, no proportional allocation exists, as by Lemma 4.3, the attained disproportionality $\max_{i \in N} DP_i(A, \mathbf{p})$ is the smallest possible, so there is no other assignment A' with $\max_{i \in N} DP_i(A', \mathbf{p}) \leq 0$. □

5 STRATEGIC MANIPULATIONS

The Minimum Disproportionality Mechanism (Algorithm 1) takes the agents' valuations as input. Ideally, we would like the agents to report their true valuations.

Formally, a *manipulation* for a mechanism $\mathcal{M}$ by an agent $i \in N$ is an untruthful report $v'_i \neq v_i$. A manipulation is *profitable* if there exists a set of reports v_{-i} from the other agents such that the agent gains a higher utility than by misreporting:

$$\exists v_{-i} : v_i(\mathcal{M}(v'_i, v_{-i})) > v_i(\mathcal{M}(v_i, v_{-i})). \tag{2}$$

A mechanism $\mathcal{M}$ is *truthful* if no agent has a profitable manipulation.

It is well-known that, even in the setting with no old apartments, no deterministic mechanism is truthful, budget-balanced and satisfies even weak fairness conditions [9, 15, 41]. In particular, if some agent i wins some apartment A_i when reporting truthfully, then a profitable manipulation for agent i is to report a slightly lower value for A_i, as in some cases the assignment will not change but the payment for agent i will increase. However, such manipulations are usually *unsafe*, as in some cases, reporting a lower value for A_i might make the algorithm choose a different assignment, so agent i would receive a worse apartment for a higher price. Therefore, one can hope that agents will not manipulate their valuations.

But with old apartments the situation is much worse: for each agent i, reporting a higher value for his old apartment o_i is both profitable and *safe*, as it has no effect on the assignment of new apartments, but it strictly increases $DP_i(A)$ and hence increases p_i by (1). Therefore, even agents will most probably manipulate their valuations.

Formally, we say that a manipulation is *safe* if it never results in a worse outcome for the manipulator—i.e., the agent weakly prefers it over truthfulness for any possible reports of the other agents:

$$\forall v_{-i} : v_i(\mathcal{M}(v'_i, v_{-i})) \geq v_i(\mathcal{M}(v_i, v_{-i})). \tag{3}$$

A mechanism $\mathcal{M}$ is *safely manipulable* if some agent has a manipulation that is both profitable and safe. Otherwise, $\mathcal{M}$ is *Risk-Avoiding Truthful (RAT)* [18, 39].

5.1 Valuations based on Characteristics

As Algorithm 1 is not even truthful, we aim to improve its strategic properties by changing the method for eliciting agents' valuations. Instead of asking each agent to evaluate each apartment, we ask them to assess the value of shared *characteristics* found in both old and new apartments, such as: directions of exposure, floor, parking, etc. This approach is inspired by the way actual real-estate appraisers compute the value of an estate. Now, if an agent wants to increase the value of his old apartment, he has to increase the value of some of its characteristics; however, these characteristics might also be present in some new apartments, which might affect the assignment in a way that decreases the manipulator's utility.

Formally, we assume that there is a fixed set T of potential apartment characteristics, such as floor level, parking availability, airflow direction, and natural light. Each agent assigns a score to each characteristic. For an apartment a and a characteristic $t \in T$, define the indicator variable $\beta_{t,a}$, such that:

$$\beta_{t,a} = \begin{cases} 1, \text{if apartment } a \text{ possesses characteristic } t \\ 0, \text{otherwise} \end{cases}$$

We consider two ways to aggregate the values of characteristics.

(1) Additive Characteristics. Here we assume that the value of an apartment is the sum of the values of its characteristics.

For each agent $i \in N$ and each characteristic $t \in T$, let $\alpha_{i,t}$ represent the value assigned by agent i to characteristic t. The valuation of apartment a for agent i is then given by

$$v_i(a) = \sum_{t \in T} \alpha_{i,t} \cdot \beta_{t,a}.$$

See [11] for an example.

(2) Multiplicative Characteristics. Here we assume that each apartment has a base price determined by its size. Each agent specifies the percentage of the apartment's base price that they attribute to each characteristic.

For each agent $i \in N$ and each characteristic $t \in T$, let $\theta_{i,t}$ be the percentage assigned by agent i to characteristic t, and let $\rho_{i,t} := 1 + \frac{\theta_{i,t}}{100}$. Note that a characteristic might have a negative effect (e.g. some people do not like apartments in high floors). In that case we set $\theta_{i,t}$ to a negative amount between -100 and 0, and get $0 < \rho_{i,t} < 1$. Equivalently, we could say θ is between 0 and 100 and define $\rho_{i,t} = 1 - \theta_{i,t}/100$.

For each apartment a, let ϕ_a denote its fixed base price, computed as the product of the apartment size in square meters, by the market price per square meter in the region. The valuation of apartment a for agent i is then given by: $v_i(a) = \phi_a \prod_{t \in T} \rho_{i,t}^{\beta_{t,a}}$. The multiplicative method is closer to the one actually used by appraisers to determine the value of an apartment in Reconstruct and Divide projects. See [11] for an example.

5.2 Manipulations of Minimum Disproportionality mechanism

Switching to characteristic-based evaluations does not guarantee that the Minimum Disproportionality mechanism is RAT. For instance, if agent i's old apartment has a unique characteristic t, increasing $v_i(t)$ safely increase their payment without affecting the assignment. Similarly, if t exists in both o_i and all new apartments, p_i still increases while the assignment remains unchanged. See [11] for details.

However, it is possible that the characteristics of the new apartment are not all known to the agents, as the valuations are elicited from the agents before the new apartments are even built. To handle this issue, we extend the definition of a safe manipulation (3) to require that the manipulation is not harmful for the agent for any combination of characteristics of the new apartments.

We show that, in this case, the Minimum Disproportionality mechanism has no safe manipulations.

For brevity, throughout this section we denote the true valuation vector $(v_1, v_2, \dots, v_n)$ as v, and the manipulated valuation vector, where only agent 1 misreports, as $v' = (v_1', v_2, \dots, v_n)$. We use $\mathbf{p}(v)$ to represent the payment vector under the valuation profile v, and let $DP(A, v)$ denote the total disproportionalitycorresponding to assignment A and valuation vector v. Let $T(a)$ denote the set of characteristics of apartment a.

Proposition 5.1. *With either multiplicative or additive characteristics, when there are $n \geq 2$ agents and the characteristics of all new apartments are unknown, the Minimum Disproportionality mechanism is risk-avoiding truthful.*

Proof sketch. Recall that the Minimum Disproportionality mechanism (Algorithm 1) computes an assignment of the new apartments that maximizes the utilitarian welfare, and a balanced payment vector $\mathbf{p}$ by (1): $p_i = DP_i(A) - \frac{DP_N(A)}{n}$.

Consider n agents with old apartments $o_1, \dots, o_n$, and suppose the new apartments are $a_1, \dots, a_n$.

Suppose agent 1 increases the values of some characteristics (in set $T_+ \subseteq T$), decreases the values of some other characteristics (in set $T_- \subseteq T \setminus T_+$), and does not change the values of the remaining characteristics (in set $T_0 := T \setminus (T_+ \cup T_-)$).

The characteristics of the o_i are known to the manipulator, and do not affect the assignment. Hence, it is sufficient to check the case that all characteristics in T_+ are in $T(o_1)$ and not in any $T(o_i)$ for $i \geq 2$, and that all characteristics in T_- are in every $T(o_i)$ for $i \geq 2$ and not in $T(o_1)$. This is clearly the best case for the manipulator, as it increases $DP_1(A)$ by the largest amount. We show that, even in that case, the manipulation is not safe. It follows that the manipulation is not safe in the cases less favorable for the manipulator.

Since the characteristics of all new apartments are unknown, it is possible that

- some new apartment (say a_1) has all the characteristics in T_+ and no characteristic from T_-;
- all other apartments a_i for $i \geq 2$ have the same set of characteristics, which contains all the characteristics in T_- and no characteristic from T_+;
- all other agents i for $i \geq 2$ have the same valuation function, v_2.

We show that, in this case, it is possible that the assignment changes. Particularly, without the manipulation agent 1 gets some a_i for $i \geq 2$ (w.l.o.g. a_2, as all these apartments are identical); and with the manipulation, agent 1 gets a_1. Note that, by the maximum-value matching definition, it suffices to show that $v_1(a_1) + v_2(a_2) <$

$v_2(a_1) + v_1(a_2)$ and $v_1'(a_1) + v_2(a_2) > v_2(a_1) + v_1'(a_2)$, as v_2 is the valuation function of *all* agents $2, \ldots, n$. These two conditions are equivalent to:

$$v_1(a_1) - v_1(a_2) < v_2(a_1) - v_2(a_2) < v_1'(a_1) - v_1'(a_2). \qquad (4)$$

We show a specific v_2 that satisfies these conditions. We also compute the changes in prices due to the manipulation, and show that, overall, agent 1 loses utility, which means that the manipulation is not safe. □

6 EXPERIMENTS

The possible non-existence of envy-free and proportional allocations has motivated us to check what envy can be attained in realistic Reconstruct and Divide projects.

We constructed a set T of 18 apartment characteristics based on professional appraisal criteria and features commonly reported in second-hand housing markets. The set was further informed by data collected from 28 apartments in real urban renewal projects in Jerusalem and Haifa, Israel, from which we identified the sizes, prices, and characteristics of both the original and newly constructed units. A complete list of characteristics is given in [11]. The base price of each apartment is computed as its size multiplied by the price per square meter.

Designing the survey required several iterations and pilot studies. We found that eliciting valuations for characteristics in context, rather than in isolation, was crucial for obtaining realistic and stable responses. To collect personalized valuations, we conducted a survey among $n = 45$ apartment owners. Each participant was asked to specify the percentage by which they believe each characteristic (positively or negatively) affects an apartment's base price.

Some respondents reported characteristic impacts that led to apartment valuations exceeding a realistic market scale (in some cases above 10 million , so we applied a normalization only when such extreme values occurred, using thresholds calibrated from real transaction data.

In the experiment, we generated n agents using a random selection from the survey-based valuations, and assigned each a random old apartment. We assumed that agents' utilities are generated by multiplicative characteristics (see Section 5), which is more similar to the method used in practice by real-estate appraisers. We computed valuations of all agents for all apartments by multiplying their values for all characteristics present in the apartment. For example, if one subject gave valuations of 1.1, 1.2 and 0.9 to the three characteristics present in the apartment, then we computed this subject's value for the entire apartment as base-value $\cdot 1.1 \cdot 1.2 \cdot 0.9$.

We made an additional experiment in which we took into account the behavioral *endowment effect*, by which people assign a higher value to an item that they own, than to the same item that they do not own [35]. For each characteristic $t \in T$, we asked each subject whether their apartment has t. We then conducted a linear regression analysis for each $t \in T$, for computing the endowment effect on t. Full results are available at [11]. A statistically-significant ownership effect ($p < 0.05$) was found only for some characteristics. Still, for robustness analysis, we applied the computed ownership correction (the coefficient of the "ownership" variable in the regression) to all properties. We plan to further examine the endowment effect in future work.

After computing the valuations, we computed an assignment maximizing utilitarian welfare via the Hungarian algorithm. Payments were computed to minimize either maximum envy (via LP) or maximum disproportionality (via the Minimum Disproportionality mechanism). We then measured both envy and disproportionality and normalized them by the average valuation across all new apartments.

We can conclude from the experiment that, even when EF is unattainable, PROP can usually be achieved. However, envy may remain high—around 20% of an apartment's value—highlighting the need for improved assignment methods (Open Question 3.8)[3].

Moreover, using survey-based valuations over apartment characteristics, we observe that the Minimum Disproportionality mechanism often attains even better-than-proportional outcomes. For details see [11].

7 CONCLUSIONS AND FUTURE WORK

We studied fairness in Reconstruct and Divide projects by considering two common fairness notions: envy-freeness and proportionality. We characterized when envy-free and proportional assignments exist. Since such assignments are not always guaranteed, we focused on minimizing envy and disproportionality through assignment and payment vectors.

We showed that the maximum attainable envy equals the maximum average mean cycle cost in the difference envy-graph. However, we could not find a polynomial-time algorithm that guarantees an optimal solution. Whether such an algorithm exists remains an open question.

On the positive side, we proposed a mechanism that minimizes maximum disproportionality. We also introduced a characteristic-based elicitation method to reduce manipulation. We also identified conditions under which the mechanism is resistant to safe manipulations. Extending these results to broader settings and identifying additional manipulation-resistant conditions remains an open direction.

We assumed that improvements are measured by differences; the model where improvements are measured by **ratios** is also conceptually appealing, but our current results for it are limited. Developing deeper theoretical insights in this model is a key area for future research.

In future work, we plan to expand our survey to include a larger and more diverse group of participants Our findings also highlight the importance of developing improved allocation methods that further reduce envy. In addition, we aim to gain a deeper understanding of the behavioral endowment effect through follow-up experiments designed to more accurately quantify its influence on perceived fairness and allocation outcomes.

[3]In this paper, we employ the multiplicative model, as it is the one commonly used in practice by property appraisers. However, based on a preliminary experiment we conducted, we are not certain that this model accurately captures the subjective valuations of apartment owners. In that experiment, we compared personalized valuations among apartment owners for pairs of unrelated apartment characteristics with the owners' personalized valuations of the individual characteristics. The results were inconclusive: for some pairs of attributes, the multiplicative model appears to describe the relationship well, whereas for others, a linear model seems to provide a better fit. We plan to further examine the applicability of the multiplicative model in future work.

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