Almost Equitable Allocations of Indivisible Goods under Externalities

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Abstract

Background: Fair division typically assumes that an agent's value depends only on its own bundle, yet many settings feature externalities where agents also derive value from others' bundles. Understanding how externalities affect standard fairness guarantees is therefore fundamental.

Objectives and Research Questions: We study equitability (EQ) and its relaxation (such as equitability upto one item (EQ1)) for indivisible goods under externalities. We ask whether such allocations exist in this model and what is the complexity of deciding or finding them.

Methods: We encounter non-existence even for small instances, and present reductions establishing NP-completeness of natural decision variants. Towards tractability, we present polynomial-time algorithms for some special cases. We also provide pseudo-polynomial and fixed-parameter tractable algorithms via dynamic programming and integer linear programming.

Results: We show that EQ1 allocations may fail to exist even for small structured instances with three agents and normalized binary valuations. For two agents, we show existence and efficient computation under normalized binary valuations. We show that deciding the existence is computationally hard under reasonable assumptions. Nevertheless, when the number of agents is a constant, we obtain a pseudo-polynomial algorithm. We also present the relation to efficient dominating sets in graph-based externalities.

Conclusions: Externalities substantially change the landscape of equitable allocations: classical existential guarantees break, and hardness arises even for highly structured instances. Our work presents a comprehensive landscape of existence and computation of such allocations.

Keywords

Fair Division, Equitable Allocations, Externalities

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1 Introduction

Allocating a set of valuable resources among individuals with preferences over them is an active area of research that has drawn the attention of economists, mathematicians, and computer scientists for several decades. The central goal is to find an allocation that satisfies various fairness criteria, such as *envy-freeness* (where no agent prefers another's bundle to their own) and *equitability* (where all agents derive the same value from their bundles).

Traditionally, the value that an agent derives depends solely on the set of resources it receives. However, real-world settings often feature *externalities*, where an agent's utility depends on the entire allocation, including items assigned to others. For instance, a friend or family member getting a club membership, with guest privileges, translates to indirect membership for oneself; allocation of the same course to a friend facilitates joint projects and resource sharing; vaccinated individuals create positive externalities for others nearby by reducing the risk of contraction.

Externalities have been studied in broader economic models, like auctions [14], supply chain management [23], welfare economics [1]. [24] mentions that externalities in fair division as important yet insufficiently studied. Recent literature on fair division of indivisible items has only explored envy-freeness and its relaxations with externalities [2, 9, 10, 22].

Equitability is an equally compelling notion of fairness, well-supported and validated by empirical evidences and a host of recent technical contributions have advanced the understanding of this fairness concept [3, 5, 6, 15, 16, 18]. In the classical setting without externalities, an EQ1 allocation is known to exist for general monotone valuations and is also efficiently computable with simple greedy procedures [11, 13]. Recently, they have also been shown to exist for non-negative valuations (which can be non-monotone) under mild assumptions [18]. However, the existence and complexity of such allocations have remained unexplored under the presence of externalities. To that end, we ask the following fundamental question:

Do EQ1 allocations exist for fair division instances with externalities?

We answer the above question in the negative. This is surprising to an extent since, in the absence of externalities, EQ1 allocations are known to exist for broad classes of valuations; however, the presence of externalities comes with non-existence even with small and structured normalized instances. We outline our contributions below.

Our Contributions

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1. We show that an EQ1 allocation may not exist even with two agents and binary valuations (Example 3.1). However, for two agents with normalized binary valuations, we show the existence and efficient computation of EQ1 allocations (Theorem 3.2).
2. For more than two agents, we show that an EQ1 allocation, unlike the classical setting, may not exist, even for binary normalized valuations and normalized externalities (Example 4.2). On the complexity side, we show that deciding the existence of EQ1 allocations such that every agent gets a minimum utility of r is NP-Complete even for normalized valuations and normalized externalities (Theorem 5.3). The hardness holds for $\{0, 1, r\}$ valuations. For binary valuations, we show that deciding the existence of EQ1 allocations such that every agent gets a minimum utility of r is NP-Complete for *nearly* normalized instances (Theorem 5.2).
   We also show that deciding the existence of EQ allocations is NP-Complete even with identical (hence, normalized) binary valuations and normalized externalities (Theorem 5.1).
3. For a constant number of agents, we show that the existence of an EQ1 allocation can be decided in pseudo-polynomial time (using a dynamic program) for general non-normalized valuations (Theorem 4.3). If the valuations are binary, then it can be done in polynomial time, again for a constant number of agents (Theorem 4.6).
4. We also present a fixed-parameter tractable algorithm to compute an EQ1 allocation, if one exists, parameterized by the number of different item-types and the number of agents (Theorem 4.8).
5. When there is an underlying graph of social network on agents, and they derive external values only from the adjacent agents in the graph, we show that an EQ1 allocation exists if the graph admits an *Efficient Dominating Set (EDS)* (Theorem 4.9). Trees and paths always admit an efficiently computable EDS, which guarantees a polynomial-time computable EQ1 allocation.

Additional Related Work

Envy-freeness and Externalities. [7] and [21] studied externalities in the divisible resource allocation. [2] formalized the externality model for EF1 allocation of indivisible items and showed with that EF1 allocations exists for three agents with binary valuations (under the assumption of no weak chores). They used a computer program to verify the existence of EF1 allocations by exhaustive search. [10] analysed the computational complexity of deciding the existence of EFX allocations under externalities and showed the decision problem is NP-Complete even for three agents or at most 6 different values for the items. Very recently, [9] showed the non-existence of EF1 allocations even with binary valuations. Additionally, they showed that EF-k allocations always exist where $k = \Theta(\sqrt{n})$.

Equitability and Its Relaxations. The recent wave of work on equitability for indivisible goods has investigated EQ1 allocations for goods [11], for chores [12], compatibility with efficiency guarantees [4], for mixed manna [17], and for valuations beyond monotone and additive [3, 6, 18]. Our work adds to this line by highlighting that externalities substantially alter the landscape.

2 Preliminaries

Model. We denote the set $\{1, 2, \ldots k\}$ as $[k]$. A fair division instance $\mathcal{I} = \langle N, M, V \rangle$ consists of a set N of n agents, a set M of m goods. An allocation $\mathcal{A} = \{A_1, A_2, \ldots A_n\}$ is a partition of the set of goods M among the set of agents N. In the fair division model with externalities, each agent $i \in N$ has a valuation function $v_i : \mathcal{A} \rightarrow \mathbb{R}^+$ that assigns a value to every allocation. The value that agent i derives when agent j receives the good g is denoted by $v_i(j, g)$. An item g is said to be a *weak chore* for an agent i if $v_i(i, g) < v_i(j, g)$. That is, i derives better value if g is assigned to some other agent than itself. In our model, we assume that there are no weak chores.

For each $i \in N$, the value i derives from an allocation $\mathcal{A}$ is given by

$$v_i(\mathcal{A}) = \sum_{j=1}^{n} \sum_{g \in A_j} v_i(j, g)$$

The valuation tuple for n agents induced by an allocation $\mathcal{A}$ is denoted by $\{v_1(\mathcal{A}), v_2(\mathcal{A}), \ldots, v_n(\mathcal{A})\}$. With a slight notation abuse, we denote the allocation $\mathcal{A}$ excluding the item $\{g\}$ by $\mathcal{A} \setminus g$. The value that agent i derives in $\mathcal{A} \setminus g$ is then $v_i(\mathcal{A} \setminus g)$.

Normalized Valuations and Externalities. We say that an instance $\mathcal{I}$ has *normalized valuations* if every agent derives a fixed total value, say K, upon receiving the grand bundle. Formally,

$$v_i(i, M) = K \quad \forall\, i \in N$$

Further, we define the external value of good g for agent i, denoted by E_i^g, as the total value agent i derives when good g is assigned to any other agent $j \neq i$. Formally, $E_g^i := \sum_{j \neq i} v_i(j, g)$. An instance $\mathcal{I}$ is said to have *normalized externalities* if the total external values across all goods is a constant for all the agents. Formally,

$$\sum_{g \in M} E_g^i = E \quad \forall\, i \in N$$

Fairness Notions. We now define the fairness notions discussed in this work. An allocation $\mathcal{A}$ is said to be Equitable (EQ) if and only if every agent derives equal utility under $\mathcal{A}$. That is, for any pair of agents $i, j \in N$, we have $v_i(\mathcal{A}) = v_j(\mathcal{A})$. Such an allocation may not even exist even in a setting without externalities; consider two agents and one good valued by both of them. Hence, approximate equitable notions have been considered in the literature. We extend these notions to the setting of externalities as follows. An allocation $\mathcal{A}$ is EQ1 iff for any $i, j \in N$, there exists a good $g \in M$ such that $v_i(\mathcal{A} \setminus g) \geq v_j(\mathcal{A} \setminus g)$. Further, an allocation $\mathcal{A}$ is EQX if an only if for any $i, j \in N$, for all $g \in M$, $v_i(\mathcal{A} \setminus g) \geq v_j(\mathcal{A} \setminus g)$.

Efficient Dominating Set (EDS). We now define an Efficient Dominating Set that we use for our algorithms and hardness results. A vertex v in a graph G dominates itself and all its neighbours. A set of vertices S in G is said to be a *dominating set* if every vertex is dominated by at least one vertex in S. Such a dominating set S is *efficient* if and only if every vertex is dominated by exactly one vertex in S. An instance $\mathcal{I} = \langle G, k \rangle$ of EDS asks if a regular graph G of degree d admits an EDS of size $k < V(G)$. This problem is known to be computationally hard even on regular graphs [8, 19].

3 Two Agents

We first present an example that shows the non-existence of an EQ1 allocation even with two agents, two items, and binary valuations.

Example 3.1. *Consider the following instance with* 2 *agents,* 2 *items and no weak chores. Under any allocation* $\mathcal{A}$*, we have* $v_1(\mathcal{A}) = 0$ *and* $v_1(\mathcal{A}) = 2$*, hence, agent* a_1 *violates EQ1. This example highlights that without normalization assumptions, EQ1 allocations do not exist even in small instances with two agents and two items, and binary valuations*

	g_1	g_2
a_1	$(0,0)$	$(0,0)$
a_2	$(1,1)$	$(1,1)$

Table 1

However, if the valuations are normalized, we show that an EQ1 allocation always exists and can be computed efficiently.

Theorem 3.2. *Given a fair division instance* $\mathcal{I} = \langle N, M, \{v_i\}_{i\in N}\rangle$ *with two agents and binary normalized valuations, an EQ1 allocation always exists and can be computed in polynomial time.*

Proof. Let a_1 and a_2 be the two agents. The goods can be divided into the following types, where the valuation vector represents $\{(v_1(a_1, g), v_1(a_2, g)), (v_2(a_1, g), v_2(a_2, g))\}$:

- $T_0 := \{(0,0),(0,0)\}$. These are the dummy items not valued by any agent.
- $T_1 := \{(1,1),(1,1)\}$. There are the items from which both the agents derive the value as well as the externality.
- $T_2 := \{(1,0),(1,1)\}$. These are the items from which both the agents derive the value, but only agent a_2 derives the externality.
- $T_3 := \{(1,1),(0,1)\}$. These are the items from which both the agents derive the value, but only agent a_1 derives the externality.
- $T_4 := \{(1,1),(0,0)\}$. These are the items valued only by a_1.
- $T_5 := \{(0,0),(1,1)\}$. These are the items valued only by a_2.
- $T_6 := \{(1,0),(0,0)\}$. These are the items valued only by a_1 without externality.
- $T_7 := \{(0,0),(0,1)\}$. These are the items valued only by a_2 without externality.
- $T_8 := \{(1,0),(0,1)\}$. These are the items from which both the agents derive the value, but no one derives the externality.

Since there are no weak chores, every item falls into one of the above types. Also, note that since the instance is normalized, we have

$$v_1(1, M) = v_2(2, M)$$

Therefore,

$$|T_1| + |T_2| + |T_3| + |T_4| + |T_6| + |T_8| = |T_1| + |T_2| + |T_3| + |T_5| + |T_7| + |T_8|$$

This implies,

$$|T_4| + |T_6| = |T_5| + |T_7| \tag{1}$$

Consider the allocation of the items as presented in Algorithm 1. We claim that the allocation $\mathcal{A}$ as output by Algorithm 1 is EQ1.

Algorithm 1 Two-Agents, Binary normalized valuations

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Input: $\mathcal{I} = \langle\{a_1, a_2\}, M, \{v_1, v_2\}\rangle$ with binary normalized valuations, no weak chores
Output: EQ1 allocation $\mathcal{A} = (A_1, A_2)$
$A_1, A_2 \leftarrow \emptyset$
for $g \in T_0 \cup T_1$ do
    Assign $g$ arbitrarily.
for $g \in T_2 \cup T_4$ do
    $A_1 = A_1 \cup \{g\}$
for $g \in T_3 \cup T_5$ do
    $A_2 = A_2 \cup \{g\}$
while $\exists g \in T_6 \cup T_7 \cup T_8$ do:
    $i \leftarrow \arg\min v_i(\mathcal{A})$
    Assign to agent $i$ its most favourite remaining item.
return $\mathcal{A} = (A_1, A_2)$
```

Claim 3.3. *The allocation* $\mathcal{A}$ *as output by Algorithm 1 is EQ1.*

Proof. First, note that the assignments for $T_0 \cup T_1 \cup T_2 \cup T_3$ contribute equally to $v_1(\mathcal{A})$ and $v_2(\mathcal{A})$. Let $\mathcal{A}'$ be the partial allocation of $\cup_{i=0}^{5} T_i$. Without loss of generality, consider $|T_4| > |T_5|$. In this case, we have $v_1(\mathcal{A}') > v_2(\mathcal{A}')$ and in particular, $v_1(\mathcal{A}') - v_2(\mathcal{A}') = |T_4| - |T_5|$. Since $|T_4| + |T_6| = |T_5| + |T_7|$ (by normalization (Equation (1))), we have $|T_6| < |T_7|$ and $|T_4| - |T_5| = |T_7| - |T_6|$. Hence, the while loop in the algorithm ensures that agent a_2 keeps receiving items it values until the disparity vanishes, in particular, a_2 receives $|T_7| - |T_6|$ that it values at 1. Consider the updated allocation $\mathcal{A}''$ after a_2 receives these $|T_7| - |T_6|$ items. Then, we have $v_1(\mathcal{A}'') = v_2(\mathcal{A}'')$. At this point, the allocation is equitable. Remaining items in $T_6 \cup T_7 \cup T_8$ yield no externalities, and hence the instance reduces to the one without externalities. As in the classical setting without externalities, the algorithm now proceeds greedily by assigning the least happy agent its most favourite remaining item. The EQ1 guarantee follows. □

This settles the claim. □

4 More than Two Agents

We now consider more than two agents. Even if the instance is normalized (but the externalities are not), the following example illustrates that there is no EQ1 allocation even with binary valuations.

Example 4.1. *Consider the following instance with* 3 *agents and* 3 *items. Under any allocation, agent* a_3 *always ends up getting the highest utility of* 3*. However, at least one of the agents from* a_1 *and* a_2 *receives a utility of at most* 1*, thereby violating EQ1 against agent* a_3*. This is primarily due to the unconstrained value that* a_3 *derives from the externalities, while the other two agents derive no external value. Notably, the valuations are normalized, yet this does not enable an EQ1 allocation.*

	g_1	g_2	g_3
a_1	$(1,0,0)$	$(1,0,0)$	$(1,0,0)$
a_2	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$
a_3	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$

Table 2

The above counter-examples motivate us to look at instances with both normalized valuations and normalized externalities.

Example 4.2 (EQ1 Non-existence). *Consider the following instance with* 4 *agents and* 2 *items. Without loss of generality, suppose* g_1 *is allocated to* a_1*. Then, exactly one agent, namely* a_2*, derives a value of one from externalities, and the other two agents,* a_3 *and* a_4 *remain poor agents. Notice that any allocation of* g_2 *only increases the inequity. Indeed, either one of the rich agents* a_1 *or* a_2 *always benefit from the allocation of* g_2 *to any agent and at least one of the poor agent (*a_3 *or* a_4 *continues to remain at the previous utility level). This implies that under any allocation, there is an agent with utility* 2 *and an agent agent with utility* 0*. This violates EQ1. Therefore, an EQ1 allocation may not exist even with normalized valuations and normalized externalities.*

	g_1	g_2
a_1	$(1,0,0,1)$	$(1,0,1,0)$
a_2	$(1,1,0,0)$	$(0,1,0,1)$
a_3	$(0,1,1,0)$	$(1,0,1,0)$
a_4	$(0,0,1,1)$	$(0,1,0,1)$

Table 3

Given the non-existence of EQ1 allocations even in structured binary instances, we anaylze the complexity of deciding the existence of such allocations. In the next sub-section, we present efficient algorithms to that end.

4.1 Constant Number of Agents

We now discuss general valuations and present dynamic-programming based algorithms for a constant number of agents.

Theorem 4.3. *Given a fair division instance* $\mathcal{I} = \langle N, M, \{v_i\}_{i \in N} \rangle$ *with externalities (not necessarily normalized), deciding the existence of an EQ1 allocation admits a pseudo-polynomial time algorithm for a constant number of agents.*

Proof. Let $\mathcal{E} := \{\mathcal{A} : \mathcal{A} \text{ is an EQ1 allocation}\}$ be the set of all possible EQ1 allocations. Let $\bar{v}(\mathcal{A}) := (v_1(A_1), \dots v_n(A_n))$ be the vector of agent values and $\bar{v}(\mathcal{E}) := \{\bar{v}(\mathcal{A}) : \mathcal{A} \in \mathcal{E}\}$. Let $v_{\max} := \max_{i,g} v_i(i,g)$ be the largest value an agent derives when a good g is allocated to itself. Then, due to no weak chores assumption, each agent's value lies between 0 and $mv_{\max}$ and hence the number of distinct valuation profiles is at most $(mv_{max})^n$. That is, $|v(\bar{\mathcal{E}})| \leq (mv_{max})^n$. Given the set $\bar{v}(\mathcal{E})$, our task is to decide if there is a value vector such that the corresponding allocation is EQ1.

Towards that end, we present a dynamic program. We maintain a table $\mathcal{T} \in m \times [m(v_{\max})^n] \times [(v_{max})^{n^2}]$ where an entry $T[t, v_1, \dots v_n, \{h_{i,j}\}_{i,j \in N}] = True$ if and only if there is an allocation of the first t items, denoted as G_t, such that agent i gets a utility of v_i and $h_{i,j} := \max_{g \in G_t}(v_i(A^{-1}(g), g) - v_j(A^{-1}(g), g))$. Then, the existence of an EQ1 allocation follows if $\mathcal{T}[m, v_1, \dots, v_n, \{h_{i,j}\}_{i,j \in N}] = True$ and for every pair of agents i and j, the following holds:

$$v_i(\mathcal{A}) - h_{i,j} \leq v_j(\mathcal{A}) \tag{2}$$

Indeed, suppose $h_{i,j} := v_i(k,g) - v_j(k,g)$ for some agent k and good $g \in A_k^t$. Then,

$$v_i(\mathcal{A}) - h_{i,j} = v_i(\mathcal{A}) - v_i(k,g) + v_j(k,g) \leq v_j(\mathcal{A}) \quad \text{(from Eq 2)}$$

This implies,

$$v_i(\mathcal{A}) - v_i(k,g) \leq v_j(\mathcal{A}) - v_j(k,g)$$
$$\Rightarrow v_i(\mathcal{A} \setminus g) \leq v_j(\mathcal{A} \setminus g)$$

and hence, j's inequity against i is resolved by removing g from agent k's bundle.

We now present the construction of the table. For the base case, we have $T[0; 0, \dots, 0; 0, \dots, 0] = True$. Inductively, we have $T[t, v_1, \dots, v_n, \{h_{i,j}\}_{i,j \in [n]}] = True$ if either

(C1) For all $i, j \in [n]$ such that $h_{i,j} = v_i(k, g_t) - v_j(k, g_t)$ for some fixed $k \in [n]$ and for some $h'_{i,j} \leq h_{i,j}$, we have $T[t-1, v_1 - v_1(k, g_t), \dots, v_n - v_n(k, g_t), \dots h'_{i,j}, ..]$ is True.

(C2) For all $i, j \in [n]$ such that $h_{i,j} > v_i(k, g_t) - v_j(k, g_t)$ for some $k \in [n]$ and $T[t-1, v_1 - v_1(k, g_t), \dots v_n - v_n(k, g_t), \dots h_{i,j}, \dots]$ is True.

We now show that the table entries are filled correctly.

Claim 4.4. *The table entry* $T[t, v_1, \dots v_n, \{h_{i,j}\}_{i,j \in [n]}] = True$ *if and only if there exists an allocation* $\mathcal{A}^t$ *of first* t *items such that* $v_i(A^t) = v_i$ *and* $\max_{g \in G_t}(v_i(A^{-1}(g), g) - v_j(A^{-1}(g), g)) = h_{i,j}$

Proof. The proof is by induction. The base case $T[0; 0, \dots, 0, 0 \dots, 0]$ is vacuously true. Suppose the claim holds true for $T[t-1, \dots]$. Let $\mathcal{A}^t = \{A_1^t, \dots, A_n^t\}$ be the allocation of the first t items such that $v_i(\mathcal{A}^t) = v_i$ and $\max_{g \in G_t}(v_i(\mathcal{A}^{-1}(g), g) - v_j(\mathcal{A}^{-1}(g), g)) = h_{i,j}$. We need to show that $T[t, v_1, \dots v_n, \{h_{i,j}\}_{i,j \in [n]}] = True$. Suppose $g_t \in A_k^t$. Let $A'_k := A_k \setminus \{g_t\}$ and $h'_{i,j} = \max_{G_{t-1}}(v_i(A^{-1}(g), g) - v_j(A^{-1}(g), g))$. Then, by induction hypothesis, we have $T[t-1, v_1 - v_1(k, g_t), \dots v_n - v_n(k, g_t), \{h'_{i,j}\}_{i,j \in [n]}]$ is True. If $h'_{i,j} > (v_i(A^{-1}(g_t), g_t) - v_j(A^{-1}(g_t), g_t))$, then $h_{i,j} = h'_{i,j}$ and condition (C2) is satisfied. If $h'_{i,j} \leq (v_i(A^{-1}(g_t), g_t) - v_j(A^{-1}(g_t), g_t))$, then condition $(C1)$ is satisfied. In either case, $T[t, v_1, \dots v_n, \{h_{i,j}\}_{i,j \in [n]}] = True$.

On the other hand, suppose $T[t, v_1, \dots v_n, \{h_{i,j}\}_{i,j \in [n]}] = True$. We need to show the existence of such an allocation. Suppose condition $(C1)$ is satisfied and let $\mathcal{A}'$ be the allocation for $T[t-1, v_1 - v_1(k, g_t) \dots v_n - v_n(k, g_t), \dots, h'_{i,j}, \dots]$. Then, let $\mathcal{A}^t$ be the allocation such that $A_k^t = A'_k \cup g_t$ and $A_i^t = A'_i$ for all agents $i \neq k$. Since $h'_{i,j} < h_{i,j}$, for the allocation $\mathcal{A}$, we get $v_i(A_i) = v_i$ and $h_{i,j} = \max_{g \in G_t}(v_i(A^{-1}(g), g) - v_j(A^{-1}(g), g)) = (v_i(A^{-1}(g_t), g_t) - v_j(A^{-1}(g_t), g_t)$. Else, if condition $(C2)$ is satisfied, let $\mathcal{A}'$ be the allocation for $T[t-1, v_1 - v_1(k, g_t) \dots v_n - v_n(k, g_t), \dots, h_{i,j}, \dots]$. Letting $A_k = A'_k \cup \{g_t\}$, since $h_{i,j} > v_1(k, g_t) - v_2(k, g_t)$, we have $h_{i,j} = \max_{g \in G_t}(v_i(A^{-1}(g), g) - v_j(A^{-1}(g), g))$ and hence,

$T[t, v_1, \dots v_n, \{h_{i,j}\}_{i,j \in [n]}]$ is verified by the allocation $\mathcal{A}^t$. This settles the claim. ◻

Given the table $\mathcal{T}$ as constructed above, we check whether any of the entries corresponding to $T[m, v_1, \dots, v_n, \{h_{i,j}\}_{i,j \in N}]$ is True. If yes, we check if $v_i - h_{i,j} \leq v_j$ for all pairs of agents i and j such that $v_i < v_j$. If it is true for any of the table entries corresponding to a complete allocation of m items, we conclude that EQ1 allocation exists and is witnessed by the allocation of the corresponding table entry. The size of the table is $m \times [m(v_{\max})^n] \times [(v_{max})^{n^2}]$ and hence, it can be constructed in psuedo-polynomial time when the number of agents is a constant. This completes the proof. ◻

We now show that if the valuations are binary, a similar approach yields a polynomial time algorithm for deciding the existence of EQ1 valuations.

4.2 Constant Number of Agents with Binary Valuations

We first prove the following characterization for EQ1 allocations with binary valuations.

Lemma 4.5. *Given fair division instances $\mathcal{I} = \langle N, M, \{v_i\}_{i \in N} \rangle$ with externalities and binary valuations, an allocation $\mathcal{A}$ is EQ1 if and only if for each agent $i \in N$, $v_i(\mathcal{A}) \in \{u, u+1\}$ for some $u \in \mathbb{Z}^+$.*

Proof. Consider an EQ1 allocation $\mathcal{A}$. Suppose $v_i(\mathcal{A}) = u$ and $v_j(\mathcal{A}) \geq u + 2$ for some $u \in \mathbb{Z}^+$. Since $\mathcal{A}$ satisfies EQ1, there must exist some $g \in M$ such that $v_i(\mathcal{A}) = u \geq v_j(\mathcal{A} \setminus \{g\}) \geq (u+2) - \max_{j'} v_j(j', g)$. Since valuation are binary, we have $\max_{j'} v_j(j', g) \leq 1$. This implies that $v_j(\mathcal{A} \setminus \{g\}) \geq u + 1$ for all $g \in M$, contradicting the fact that $\mathcal{A}$ is EQ1. Therefore, $v_j(\mathcal{A}) \leq u + 1$ for all $j \in N$.

On the other hand, suppose $v_i(\mathcal{A}) \in \{u, u+1\}$ for all $i \in N$. Let $v_i(\mathcal{A}) = u$ and $v_j(\mathcal{A}) = u + 1$. Then, there is an item g such that j derives a value of 1 from the allocation of g (to any agent) but i derives a value of 0. Hence, $u = v_i(\mathcal{A}) \geq v_j(\mathcal{A} \setminus \{g\}) = (u+1) - 1 = u$. This concludes that $\mathcal{A}$ is EQ1. ◻

We are now ready to present the polynomial time algorithm for constant number of agents and binary valuations.

Theorem 4.6. *Given fair division instances $\mathcal{I} = \langle N, M, \{v_i\}_{i \in N} \rangle$ with externalities and binary valuations (not necessarily normalized), deciding the existence of an EQ1 allocation admits a polynomial-time algorithm for a constant number of agents.*

Proof. Let $\mathcal{E} := \{\mathcal{A} : \mathcal{A} \text{ is an EQ1 allocation}\}$ be the set of all possible EQ1 allocations. Let $\bar{v}(\mathcal{A}) := (v_1(A_1), \dots v_n(A_n))$ be the vector of agent values and $\bar{v}(\mathcal{E}) := \{\bar{v}(\mathcal{A}) : \mathcal{A} \in \mathcal{E}\}$. Let $v_{\max}$ be the largest value an agent derives when a good g is allocated to itself. Since valluations are binary, we have $v_{\max} = 1$. Then, each agent's value lies between 0 and $mv_{\max} = m$ and hence the number of distinct valuation profiles is at most m^n. That is, $|v(\bar{\mathcal{E}})| \leq m^n$. Given the set $\bar{v}(\mathcal{E})$, our task is to decide if there is a value vector such that the corresponding allocation is EQ1. Towards that end, we present a dynamic program. We maintain a table $\mathcal{T} \in m \times m^n$ where an entry $T[t, v_1, v_2 \dots v_n] = True$ if an only if there is an allocation of first t items $G_t := \{g_1, g_2, \dots g_k\}$ such that an agent i derives a utility of v_i. The states $T[m, v_1, v_2 \dots v_n]$ correspond to the final allocations. By Lemma 4.5, there is an EQ1 allocation if and only if $v_i \in \{u, u+1\}$ for all $i \in N$, where $u \in \mathbb{Z}^+$. The total number of table entries are $O(m^{n+1})$.

We now present the bottom-up construction of the table. The base case $T[0; 0, \dots 0]$ is vacuously true. A general state $T[t, v_1, \dots, v_n]$ is true if $T[t-1, v_1 - v_1(k, g_t), \dots v_n - v_n(k, g_t)]$ is True for any agent $k \in N$. That is,

$$T[t, v_1, \dots, v_n] = \vee_{k \in N} T[t-1, v_1 - v_1(k, g_t), \dots v_n - v_n(k, g_t)]$$

We now show that the table entries are filled correctly.

Claim 4.7. *The table entry $T[t, v_1, \dots, v_n] = True$ is and only there exists an allocation $\mathcal{A}^t$ of first t items such that $v_i(\mathcal{A}^t) = v_i$.*

Proof. The proof follows by induction. The base case $T[0; 0, \dots 0]$ is vacuously true. Suppose the claim holds true for $T[t-1, \dots]$. Let $\mathcal{A}^t$ be the allocation of the first t items such that $v_i(\mathcal{A}^t) = v_i$. Let k be the agent who gets the item g_t under $\mathcal{A}^t$. Consider the allocation $\mathcal{A}'$ such that $\mathcal{A}'_k := \mathcal{A}^t_k \setminus g_t$. Then, $\mathcal{A}'$ is an allocation of $t-1$ items such that $v_i(\mathcal{A}^i) = v_i - v_i(k, g_t)$. Hence, by induction hypothesis, $T[t-1, v_1 - v_1(k, g_t), \dots, v_n - v_n(k, g_t)]$ is True. Therefore, by construction, $T[t, v_1, \dots v_n]$ is True.

On the other hand, suppose $T[t, v_1, \dots, v_n]$ is true. We need to show the existence of such an allocation. Since $T[t, v_1, \dots, v_n]$ is true, we have $\vee_{k \in N} T[t-1, v_1 - v_1(k, g_t), \dots v_n - v_n(k, g_t)]$ is true. Hence, there is an agent $k \in N$ such that $T[t-1, v_1 - v_1(k, g_t), \dots, v_n - v_n(k, g_t)]$ is true. Again by induction hypothesis, there exists an allocation of $\mathcal{A}'$ first $t-1$ items such that $v_i(\mathcal{A}') = v_i - v_i(k, g_t)$. Then, consider the allocation $\mathcal{A}^t$ such that $A^t_k := A'_k \cup \{g_t\}$ and $A^t_i := A'_i$ for all $i \neq k$. Then, $v_i(\mathcal{A}^t) = v_i$ for all $i \in N$ and this settles the claim. ◻

Given the table $\mathcal{T}$ as constructed above, we check if any of the table entries corresponding to $T[m, v_1 \dots v_n]$ is true. If yes, we check if $v_i \in \{u, u+1\}$, and if yes, we have that the corresponding alloctaion $\mathcal{A}$ is EQ1 (by Lemma 4.5). ◻

4.3 An FPT Algorithm

Theorem 4.8. *Given a fair division instance $\mathcal{I} = \langle N, M, \{v_i\}_{i \in N} \rangle$ with externalities, computing an EQ1 allocation admits fixed-parameter tractable algorithm parameterized by the number of different item-types and the number of agents.*

Proof. Let τ be the total number of item-types and $T = \{T_1, T_2, \dots, T_\tau\}$ be the partition of the items according to their types. Let n_{T_i} be the number of items in type T_i. The bundle-type captures the different item-types with at least one representative in the bundle. Since there are τ types of items, there are at most 2^τ bundle-types. For each agent, we guess its bundle-type. These amount to 2^{τ^n} such guesses. For each such guess, we also construct a verification ILP that verifies if the guess satisfies EQ1. To that end, let $B(i)$ be the set of item-types present in agent $i's$ bundle according to the guess. Let $v_i(j, t)$ denote the value that agent i derives when an item of type t is assigned to agent j. We compute for each agent i, the item-type t', and agent i', such that the removal of t' from $i's$ bundles witnesses the largest decrease in the value that i derives from the allocation.

We define the variables $x_{t,i}$ denoting the number of items of type t assigned to agent i. The constraint are as follows:

$$\sum_{i\in N} x_{t,i} = n_t \quad \forall t \in T \tag{3}$$

$$x_{t,i} = \begin{cases} \geq 1 & \text{if } t \in B(i) \\ = 0 & \text{if } t \notin B(i) \end{cases} \qquad \forall i \in N, \forall t \in T \tag{4}$$

$$\begin{aligned} \sum_{t\in T}(x_{t,i}v_i(i,t) + \sum_{j\neq i} x_{t,j}v_i(j,t)) \geq & \sum_{t\in T}(x_{t,j}v_j(j,t)+ \\ & \sum_{i\neq j} x_{t,i}v_j(i,t))- \\ & \max_{(j',t'\in B(j'))}(v_j(j',t') - v_i(j',t')) \\ & (\forall i,j \in N) \end{aligned} \tag{5}$$

The constraints in Equation (3) ensures that all the items are allocated. The constraints in Equation (4) ensure that the allocation correspond to the guess in question. Finally, the constraint in Equation (5) verify that the allocation is EQ1. The total number of variables in the ILP is equal to the number of agents times the number of item-types ($O(n \cdot \tau)$). [20] showed that given an integer linear program in d variables, it can be solved in time $f(d) \cdot \text{poly}(L)$, where L denotes the input size and f depends only on d. Therefore, this concludes the proof. □

4.4 EQ1 Allocations and Efficient Dominating Set

Theorem 4.9. *Given a fair division instance $\mathcal{I} : \langle N, M, \{v_i\}_i \in N\rangle$ with externalities arising from an underlying graph G on the agents, $\mathcal{I}$ admits an EQ1 allocation if G admits an Efficient Dominating Set.*

Proof. Suppose the G on the set of agents admits an EDS, say S. Then, every vertex in $V(G)$ is dominated by exactly one vertex in $S = \{s_1, s_2, \ldots s_k\}$. Consider an EQ1 allocation $\mathcal{A}'$ of items in M (without considering the externalities) to the agents corresponding to the vertices in S. (A classical EQ1 allocation always exists; give the least happy agent its most favourite available item.) Extend this to an allocation $\mathcal{A}$ among all the n agents as follows:

$$A_i = \begin{cases} A'_i, & \text{if } i : s_i \in S \\ \emptyset, & \text{otherwise} \end{cases}$$

We claim that $\mathcal{A}$ is EQ1 for the instance $\mathcal{I}$ with externalities. Indeed, since $\mathcal{A}'$ is EQ1 for all the agents corresponding to S, we have that $v_i(A'_i) \geq v_j(A'_j \setminus g)$ for some $g \in M$. Since S is an EDS and hence, is an independent set, $v_i(j,g) = 0$ for all $i, j : s_i, s_i \in S$. Therefore, we have $v_i(\mathcal{A}) = v_j(\mathcal{A} \setminus g)$ for all $i, j : s_i, s_j \in S$. All the agents i' such that $s_{i'} \notin S$ are adjacent to exactly one vertex in S, say i, again by the definition of EDS. Therefore, $v_{i'}(\mathcal{A}) = v_i(\mathcal{A}) \geq v_j(\mathcal{A} \setminus g)$ for all $j \neq i : s_j$. This settles the claim that $\mathcal{A}$ is EQ1. □

5 Hardness Results

We now analyze the computational complexity of deciding the existence of EQ and EQ1 allocations. We first show that deciding whether an EQ allocation exists is NP-Complete even with identical binary valuations and normalized externalities.

Theorem 5.1. *Given fair division instances $\mathcal{I} = \langle N, M, \{v_i\}_{i\in N}\rangle$, deciding the existence of an EQ allocation is NP-Complete even with identical (hence, normalized) $\{0,1\}$ valuations and normalized externalities.*

Proof. We present a reduction from EDS. This problem is known to be computationally hard even on regular graphs [8, 19].

Given an instance $\mathcal{I}' = \langle G, k\rangle$ of EDS, we construct the allocation instance $\mathcal{I}$ as follows. We construct $m = k$ items and $n = |V(G)|$ agents. Every agent i values any item $\{g_i\}_{i\in[k]}$ at 1 and derives an externality of 1 if any item is allocated to an agent j such that $(ij) \in E(G)$. That is, we have $v_i(i,g) = 1$ for all $i \in N, g \in M$ and, for all j such that $(ij) \in E(G)$, we have $v_i(j,g) = 1$ for all $i \in N$ and $g \in M$. This completes the construction.

We now argue the equivalence.

Forward Direction. Suppose $\mathcal{I}'$ is a yes instance and let S be the size-k EDS of G. Then consider the allocation $\mathcal{A}$ which allocates exactly one item to the agents corresponding to the vertices in S. Then, for all agents i such that $i \in S$, we have $v_i(\mathcal{A}) = 1$. Also, since S is an EDS, any vertex not in S is adjacent to exactly one vertex in S. So, for all agents j such that $j \notin S$, $v_j(i,g) = 1$ for exactly one agent $i \in S$. Therefore, $v_j(\mathcal{A}) = 1$. Hence, $\mathcal{A}$ is an EQ allocation and $\mathcal{I}$ is a yes instance.

Reverse Direction. Suppose $\mathcal{I}$ is a yes-instance and $\mathcal{A}$ is an EQ allocation. We first claim that no agent can receive two items under $\mathcal{A}$. Suppose some agent i got two items under $\mathcal{A}$. Then, $v_i(\mathcal{A}) \geq 2$. Since there are only k items and at least one agent ends up getting more than one item, at most $k-1$ agents can receive the actual items, and the rest of them derive their utility from the external value. Since G is a regular graph of degree d, the $k-1$ vertices can be adjacent to at most $(k-1)d$ vertices. However, the total number of vertices in G is $k(d+1)$ (the existence of size-k EDS ensures that the closed neighbourhood of vertices in the EDS forms a partition of $V(G)$). This implies that only $(k-1)+(k-1)d = (k-1)(d+1)$ agents get a non-zero utility under $\mathcal{A}$, leaving behind $k(d+1)-(k-1)(d+1) = d+1$ agents with 0 utility. Therefore, for $\mathcal{A}$ to be an EQ allocation, every agent must get at most one item.

Let S be the set of k agents who receive exactly one item under $\mathcal{A}$. We claim that S forms an EDS of G. We have $v_i(\mathcal{A}) \geq 1$ for all $i \in S$. Since $\mathcal{A}$ is an EQ allocation, we must have $v_j(\mathcal{A}) \geq 1$ for all $j \notin S$. There are kd many agents outside S, and none of them receive any item under $\mathcal{A}$, they must have derived their utility as an external value. This implies that every $j \notin S$ is adjacent to some vertex in S. Since each of the k vertices in S can only dominate d many vertices because of the degree bound, and each of the kd vertices outside S must be dominated for them to receive a non-zero utility, it must be the case that S is an independent set. This implies that $v_i(\mathcal{A}) = 1$ for all $i \in S$. Since $\mathcal{A}$ is EQ, we get $v_j(\mathcal{A}) = 1$ for al $j \notin S$, consequently every agent not in S is adjacent to exactly one agent in S. Hence, S forms an EDS of size k, thereby $\mathcal{I}'$ is a yes instance. □

We now show that deciding the existence of EQ1 allocations such that every agent gets a minimum utility of r is hard even for (nearly normalized) binary valuations.

Theorem 5.2. *Given fair division instances $\mathcal{I} = \langle N, M, \{v_i\}_{i \in N} \rangle$, deciding the existence of an EQ1 allocation with a minimum utility of r is hard even with nearly normalized $\{0,1\}$ valuations.*

Proof. We present a reduction from Equitable 3 Coloring wherein given an instance $\mathcal{I}' = \langle G, k \rangle$, with $|V(G)| = 3r$, the problem is to decide if there is a partition of the vertex set $V(G)$ into 3 color classes (independent sets) $\{V_1, V_2, V_3\}$ such that $|V_1| = |V_2| = |V_3| = r$. We construct the fair division instance as follows. We create $n(= 3r)$ *vertex-items* $\{g_v\}_{v \in V(G)}$, and $3mr$ *edge-items* $\{g_{uv}^{c_j}\}$ for $uv \in E(G)$, $c \in \{x, y, z\}$ and $j \in [r]$.

We create the following agents.

1. 3 *class-agents*, namely, $\{a^x, a^y, a^z\}$, that correspond to the three color classes $\{x, y, z\}$. These agents value all the $3r$ vertex items at 1 each. In aggregate, the class-agents value the entire bundle of goods at $3r$ and derive 0 externality from any assignment.
2. $3m$ *edge-agents* for each pair of an edge and a color class, namely, a_{uv}^c for $uv \in E(G)$ and $c \in \{x, y, z\}$. These agents value all the r copies of the corresponding edge-item in their color-class at 1 each. In particular, a_{uv}^x values r items $\{g_{uv}^{x_j}\}_{j \in [r]}$. For each such edge-item that they value, they also derive an externality of 1 regardless of which agent the item is allocated to.

   Every agent a_{uv}^c also values the vertex-items corresponding g_u and g_v to their end points. The agent a_{uv}^c derives an externality of 1 each if g_u and g_v is allocated to a^c. In aggregate, a_{uv}^c values $r + 2$ items and derives a total externality of $(n-1)r + 2$.

These add up to a total of $3m + 3$ agents and $3r + 3mr$ items. Note that every agent values either exactly $3r$ or $r + 2$ items, hence the instance is *nearly* normalized. Additionally, the vertex agents derive a total externality of 0 each, while the edge agents derive a total externality of $(n-1)r + 2$. The completes the construction.

We now argue the equivalence.

Forward Direction. Suppose Equitable 3-coloring is a yes-instance and there is a partition of the vertices into 3 independent sets V_1, V_2, V_3 such that $|V_i| = r$. Then, consider the following allocation A. The 3 class agents $\{a^x, a^y, a^z\}$ get the 3 partitions $\{V_1, V_2, V_3\}$ respectively, and each edge agent a_{uv}^c gets r edge items. Then, it is easy to see that every agent either gets a utility of r or $r + 1$. Note that since V_i are independent sets, no end-points of an edge are allocated to the same agent; consequently, the utility of every edge agent is at most $r + 1$. This concludes that A is an EQ1 allocation with minimum utility r.

Reverse Direction. Suppose there is an EQ1 allocation such that the minimum utility is r. Then, note that all the vertex items must be allocated to the class-agents, since they do not value anything else. This forces the $3r$ vertex items to be equally divided among the 3 class-agents. What remains are the edge-items, which can be allocated to anyone, but due to the externalities, the edge-agents will end up getting a utility of at least r from the edge-item regardless of which agent receives them. An edge agent, say a_{uv}^x can also receive a utility of 2 if two vertex-items $\{u, v\}$ such that $(uv) \in E(G)$ end up with a class agent, say a^x. But then the total utility of a_{uv}^x becomes at least $r + 2$, which violates the EQ1 constraint, since the maximum utility can be at most $r + 1$. Therefore, no two end-points of an edge are allocated to the same agent under any EQ1 allocation, which implies that there is a vertex partition of $3r$ vertices into 3 independent sets, each of size r. This implies that Equitable 3 Coloring is a yes-instance. The theorem stands proved. □

Going slightly beyond binary valuations, we now show that deciding the existence of EQ1 allocations such that every agent gets a utility of at least r is hard even with $\{0, 1, r\}$ normalized valuations and normalized externalities.

Theorem 5.3. *Given fair division instances $\mathcal{I} = \langle N, M, \{v_i\}_{i \in N} \rangle$, deciding the existence of an EQ1 allocation with a minimum utility of r is hard even with $\{0, 1, r\}$-normalized valuations and normalized externalities.*

Proof. We present a reduction from Equitable 3 Coloring wherein given an instance $\mathcal{I}' = \langle G, k \rangle$, with $|V(G)| = 3r$, the problem is to decide if there is a partition of the vertex set $V(G)$ into 3 color classes $\{V_1, V_2, V_3\}$ such that $|V_1| = |V_2| = |V_3| = r$. We construct the fair division instance as follows. We create $n(= 3r)$ *vertex-items* $\{g_v\}_{v \in V(G)}$, $3m$ *edge-items* $\{g_{uv}^c\}$ for $uv \in E(G)$, $c \in \{x, y, z\}$ and 3 *dummy-items* $\{g_{d_1}, g_{d_2}, g_{d_3}\}$.

We create the following agents.

1. 3 *class-agents*, namely, $\{a^x, a^y, a^z\}$, that correspond to the three color classes $\{x, y, z\}$. These agents value all the $3r$ vertex items at 1 each. They also value two dummy items g_{d_1} and g_{d_2}. They derive an externality of 1 each if g_{d_1} and g_{d_2} are allocated to a fixed edge agent, say, a_{uv}^x for edge $(uv) \in E(G)$. In aggregate, $\{a^c\}_{c \in \{x,y,z\}}$ values the entire set of items at $3r + 2$ and derives a total externality of 2.
2. $3m$ *edge-agents* for each pair of an edge and a color class, namely, a_{uv}^c for $uv \in E(G)$ and $c \in \{x, y, z\}$. These agents value all the corresponding 3 edge-items at r each. In particular, a_{uv}^x values $\{g_{uv}^c\}$ where $c \in \{x, y, z\}$. Every agent a_{uv}^c also values the vertex-items corresponding g_u and g_v to their end points at 1 each. The agent a_{uv}^c derives an externality of 1 each if g_u and g_v are allocated to a^c. In aggregate, a_{uv}^c values all the entire set of items at $3r + 2$ and derives a total externality of 2.
3. 3 *dummy-agents* $\{a_{d_i}\}_{i \in [3]}$. The agent a_{d_i} values all the g_{d_i} at r while the remaining two dummy items at $r + 1$ each and drives an externality of 1 each when g_{d_i} is allocated to either of the other two dummy agents. In aggregate, $\{a_{d_i}\}_{i \in [3]}$ values all the entire set of items at $3r + 2$ and derives a total externality of 2.

These add up to a total of $3m + 6$ agents and $3r + 3m + 3$ items. Note that every agent values exactly $3r + 2$ items, derives a total externality of 2 each. Hence, the instance has normalized valuations and normalized externalities. This completes the construction.

We now argue the equivalence.

Forward Direction. Suppose Equitable 3-coloring is a yes-instance and there is a partition of the vertices into 3 independent sets V_1, V_2, V_3 such that $|V_i| = r$. Then, consider the following allocation A. The 3 class agents $\{a^x, a^y, a^z\}$ get the 3 partitions $\{V_1, V_2, V_3\}$ respectively, each edge agent a^c_{uv} gets the edge-item g^c_{uv}, the dummy agents a_{d_i} gets the dummy-item g_{d_i}. Then, it is easy to see that every agent either gets a utility of r or $r + 1$. Note that since V_i are independent sets, no end-points of an edge are allocated to the same agent; consequently, the utility of every edge agent is at most $r + 1$. This concludes that A is an EQ1 allocation with minimum utility r.

Reverse Direction. Suppose there is an EQ1 allocation such that the minimum utility is r. Then, note that all the dummy items must be allocated to the dummy agents, since they do not value anything else. This forces the $3r$ vertex items to be equally divided among the 3 vertex agents. The 3 edge-items $\{g^c_{uv}\}_{c \in \{x,y,z\}}$ must be allocated equally among the 3 edge-agents $\{a^c_{uv}\}_{c \in \{x,y,z\}}$, otherwise, the value of the edge-agents remain strictly less than r. Additionally, an edge agent, say a^x_{uv} can also receive a utility of 2 if the two vertex-items $\{g_u, g_v\}$ such that $(uv) \in E(G)$ end up with a class agent, say a^x. But then the total utility of a^x_{uv} becomes at least $r + 2$, which violates the EQ1 constraint, since the maximum utility can be at most $r + 1$. Therefore, no two end-points of an edge are allocated to the same agent under any EQ1 allocation, which implies that there is a vertex partition of $3r$ vertices into 3 independent sets, each of size r. This implies that Equitable 3 Coloring is a yes-instance. The theorem stands proved. □

6 Conclusion

In this work, we explore the existence and computational complexity of EQ1 allocations in fair division problems with externalities. We demonstrate that the presence of externalities significantly alters the landscape of equitable allocations, leading to non-existence even for small and normalized instances. We also show the computational hardness of deciding the existence of EQ1 allocations with a minimum utility guarantee to every agent. Towards tractability, we present an efficient algorithm that outputs an EQ1 allocation for two agents and binary normalized valuations. We also present pseudo-polynomial and fixed-parameter tractable algorithms for specific settings. Our work highlights the challenges posed by externalities in fair division and opens avenues for further research into efficient algorithms and approximation techniques in such settings. The complexity of deciding the existence of EQ1 allocations for binary normalized instances with normalized externalities remains open. One immediate interesting direction is to explore the existence of EQ-k allocations, where the inequity can be eliminated by the removal of at most k items.

References

[1] Robert U. Ayres and Allen V. Kneese. Production, consumption, and externalities. *The American Economic Review*, 59(3):282–297, 1969. ISSN 00028282.

[2] Haris Aziz, Warut Suksompong, Zhaohong Sun, and Toby Walsh. Fairness concepts for indivisible items with externalities. *Proceedings of the AAAI Conference on Artificial Intelligence*, 37(5):5472–5480, Jun. 2023.

[3] Siddharth Barman, Umang Bhaskar, Yeshwant Pandit, and Soumyajit Pyne. Nearly equitable allocations beyond additivity and monotonicity. AAAI'24/IAAI'24/EAAI'24. AAAI Press, 2024. ISBN 978-1-57735-887-9.

[4] Umang Bhaskar, Neeldhara Misra, Aditi Sethia, and Rohit Vaish. The price of equity with binary valuations and few agent types. In *Algorithmic Game Theory: 16th International Symposium, SAGT 2023, Egham, UK, September 4–7, 2023, Proceedings*, page 271–289, Berlin, Heidelberg, 2023. Springer-Verlag. ISBN 978-3-031-43253-8.

[5] Umang Bhaskar, Vishwa Prakash HV, Aditi Sethia, and Rakshitha. Best of both worlds guarantees for equitable allocations, 2025. URL https://arxiv.org/abs/2505.05809.

[6] Vittorio Bilò, Martin Loebl, and Cosimo Vinci. Approximately envy-free and equitable allocations of indivisible items for non-monotone valuations, 2025. URL https://arxiv.org/abs/2503.05695.

[7] Simina Brânzei, Ariel D. Procaccia, and Jie Zhang. Externalities in cake cutting. In *Proceedings of the Twenty-Third International Joint Conference on Artificial Intelligence*, IJCAI '13, page 55–61. AAAI Press, 2013. ISBN 9781577356332.

[8] Domingos M. Cardoso, J. Orestes Cerdeira, Charles Delorme, and Pedro C. Silva. Efficient edge domination in regular graphs. *Discrete Applied Mathematics*, 156 (15):3060–3065, 2008. ISSN 0166-218X.

[9] Frank Connor, Max Dupré la Tour, Vishnu V. Narayan, and Šimon Schierreich. Tight asymptotic bounds for fair division with externalities, 2026. URL https://arxiv.org/abs/2601.13287.

[10] Argyrios Deligkas, Eduard Eiben, Viktoriia Korchemna, and Šimon Schierreich. The complexity of fair division of indivisible items with externalities. In *Proceedings of the Thirty-Eighth AAAI Conference on Artificial Intelligence and Thirty-Sixth Conference on Innovative Applications of Artificial Intelligence and Fourteenth Symposium on Educational Advances in Artificial Intelligence*, AAAI'24/IAAI'24/EAAI'24. AAAI Press, 2024. ISBN 978-1-57735-887-9. doi:10.1609/aaai.v38i9.28822. URL https://doi.org/10.1609/aaai.v38i9.28822.

[11] Rupert Freeman, Sujoy Sikdar, Rohit Vaish, and Lirong Xia. Equitable allocations of indivisible goods. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence*, IJCAI'19, page 280–286. AAAI Press, 2019.

[12] Rupert Freeman, Sujoy Sikdar, Rohit Vaish, and Lirong Xia. Equitable allocations of indivisible chores. In *Proceedings of the 19th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 384–392, 2020.

[13] Laurent Gourvès, Jérôme Monnot, and Lydia Tlilane. Near fairness in matroids. In *Proceedings of the Twenty-First European Conference on Artificial Intelligence*, ECAI'14, pages 393–398, NLD, 2014. IOS Press. ISBN 978-1-61499-418-3.

[14] Nima Haghpanah, Nicole Immorlica, Vahab Mirrokni, and Kamesh Munagala. Optimal auctions with positive network externalities. *ACM Trans. Econ. Comput.*, 1(2), May 2013. ISSN 2167-8375.

[15] Dorothea K. Herreiner and Clemens Puppe. Inequality aversion and efficiency with ordinal and cardinal social preferences—an experimental study. *Journal of Economic Behavior and Organization*, 76(2), 2010.

[16] Dorothea K. Herreiner and Clemens D. Puppe. Envy freeness in experimental fair division problems. *Theory and Decision*, 67(1):65–100, 07 2009.

[17] Hadi Hosseini and Aditi Sethia. Equitable allocations of mixtures of goods and chores. *arXiv preprint arXiv:2501.06799*, 2025.

[18] Hadi Hosseini, Vishwa Prakash HV, Aditi Sethia, and Jatin Yadav. The landscape of almost equitable allocations, 2025. URL https://arxiv.org/abs/2511.07395.

[19] Jan Kratochvíl. Regular codes in regular graphs are difficult. *Discrete Mathematics*, 133(1):191–205, 1994. ISSN 0012-365X. doi:https://doi.org/10.1016/0012-365X(94)90026-4. URL https://www.sciencedirect.com/science/article/pii/0012365X94900264.

[20] Hendrik W. Lenstra Jr. Integer programming with a fixed number of variables. *Math. Oper. Res.*, 8(4):538–548, 1983. doi:10.1287/moor.8.4.538. URL https://doi.org/10.1287/moor.8.4.538.

[21] Minming Li, Jialin Zhang, and Qiang Zhang. Truthful cake cutting mechanisms with externalities: do not make them care for others too much! In *Proceedings of the 24th International Conference on Artificial Intelligence*, IJCAI'15, page 589–595. AAAI Press, 2015. ISBN 9781577357384.

[22] Shaily Mishra, Manisha Padala, and Sujit Gujar. Fair allocation with special externalities. In *PRICAI 2022: Trends in Artificial Intelligence: 19th Pacific Rim International Conference on Artificial Intelligence, PRICAI 2022, Shanghai, China, November 10–13, 2022, Proceedings, Part I*, page 3–16, Berlin, Heidelberg, 2022. Springer-Verlag. ISBN 978-3-031-20861-4.

[23] Serguei Netessine and Fuqiang Zhang. Positive vs. negative externalities in inventory management: Implications for supply chain design. *Manuf. Serv. Oper. Manag.*, 7:58–73, 2005. URL https://api.semanticscholar.org/CorpusID:12051913.

[24] Masoud Seddighin, Hamed Saleh, and Mohammad Ghodsi. Externalities and fairness. In *The World Wide Web Conference*, WWW '19, page 538–548, New York, NY, USA, 2019. Association for Computing Machinery. ISBN 9781450366748.